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Geometry of Cyclic and Anticylic Algebras

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Abstract

The article deals with spaces the geometry of which is defined by cyclic and anticyclic algebras. Arbitrary multiplicative function is taken as a fundamental form. Motions are given as linear transformation preserving given multiplicative function.

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Real cyclic and anticyclic algebras $\mathbf{R}^+_m(\boldsymbol{\varepsilon})$, $\mathbf{R}^-_m(\boldsymbol{\varepsilon})$ with generator $\boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}^m = \pm 1$, may be considered as a generalization of the algebra of dual real numbers or the field of complex numbers and they are isomorphic to the following direct sums (see [1]):

$$\begin{aligned} \mathbf{R}^+_m(\boldsymbol{\varepsilon}) &\cong \begin{cases} \mathbf{R}\oplus\mathbf{R}\oplus\underbrace{\mathbf{C}\oplus\ldots\oplus\mathbf{C}}_{k-1}, m=2k \\ \mathbf{R}\oplus\overbrace{\mathbf{C}\oplus\ldots\oplus\mathbf{C}}^{k}, m=2k+1 \end{cases} \\ \mathbf{R}^-_m(\boldsymbol{\varepsilon}) &\cong \begin{cases} \underbrace{\mathbf{C}\oplus\ldots\oplus\mathbf{C}}_{k}, m=2k \\ \mathbf{R}\oplus\overbrace{\mathbf{C}\oplus\ldots\oplus\mathbf{C}}^{k}, m=2k+1 \end{cases} \end{aligned} \tag{1}$$

These isomorphisms determine the existence of certain functions $F\colon \mathbf{R}^\pm_m(\boldsymbol{\varepsilon}) \to \mathbf{R}$, such that $F(\boldsymbol{x}\cdot\boldsymbol{y}) = F(\boldsymbol{x})F(\boldsymbol{y})$ for every $\boldsymbol{x},\boldsymbol{y} \in \mathbf{R}^\pm_m(\boldsymbol{\varepsilon})$. This property is called composition property (see [2]).

As $\mathbf{R}_m^{\pm}(\boldsymbol{\varepsilon}) \subset \mathbf{C}_m(\boldsymbol{\varepsilon})$, the composition property of algebras $\mathbf{R}_m^{\pm}(\boldsymbol{\varepsilon})$ as well as the type of multiplicative functions $F\colon \mathbf{R}_m^{\pm}(\boldsymbol{\varepsilon}) \to \mathbf{R}$ follows from composition property of complex cyclic algebras $\mathbf{C}_m(\boldsymbol{\varepsilon})$. To prove the composition property of algebras $\mathbf{C}_m(\boldsymbol{\varepsilon})$ we consider a resolvent $\boldsymbol{x}(\alpha_m^k)$ of an arbitrary element (see [3]):

If

$$\boldsymbol{x} = x_0 + x_1\boldsymbol{\varepsilon} + x_2\boldsymbol{\varepsilon}^2 + \ldots + x_{m-1}\boldsymbol{\varepsilon}^{m-1} \in \mathbf{C}_m(\boldsymbol{\varepsilon}),$$

then

$$\boldsymbol{x}(\alpha_m^k) = x_0 + \alpha_m^k x_1\boldsymbol{\varepsilon} + \alpha_m^{2k} x_2\boldsymbol{\varepsilon}^2 + \ldots + \alpha_m^k x_{m-1}\boldsymbol{\varepsilon}^{k(m-1)} \in \mathbf{C}_m(\boldsymbol{\varepsilon}),$$

where $\alpha_m = \cos(2\pi/m) + \mathrm{i}\sin(2\pi/m)$. The mapping $\hat{\boldsymbol{\alpha}}_m\colon \mathbf{C}_m(\boldsymbol{\varepsilon}) \to \mathbf{C}_m(\boldsymbol{\varepsilon})$, such that $\hat{\boldsymbol{\alpha}}_m(\boldsymbol{x}) = \boldsymbol{x}(\alpha_m)$, is called a *resolvent mapping* and it fulfils the following identities:

$$\hat{\boldsymbol{\alpha}}_m(\boldsymbol{x}\cdot\boldsymbol{y}) = \hat{\boldsymbol{\alpha}}_m(\boldsymbol{x})\cdot\hat{\boldsymbol{\alpha}}_m(\boldsymbol{y}) \quad \text{and} \quad \hat{\boldsymbol{\alpha}}_m^m(\boldsymbol{x}) = \boldsymbol{x}, \quad \text{for any } \boldsymbol{x},\boldsymbol{y} \in \mathbf{C}_m(\boldsymbol{\varepsilon}).$$

Now, let us consider a function $\Delta\colon \mathbf{C}_m(\boldsymbol{\varepsilon}) \to \mathbf{C}_m(\boldsymbol{\varepsilon})$ which an arbitrary $\boldsymbol{a} \in \mathbf{C}_m(\boldsymbol{\varepsilon})$ maps to the value of the determinant of the following system of equations:

$$\begin{gathered}
a_0x_0 + a_{m-1}x_1 + a_{m-2}x_2 + \cdots + a_1x_{m-1} = b_0\\
a_1x_0 + a_0x_1 + a_{m-1}x_2 + \cdots + a_2x_{m-1} = b_1\\
a_2x_0 + a_1x_1 + a_0x_2 + \cdots + a_3x_{m-1} = b_2\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
a_{m-1}x_0 + a_{m-2}x_1 + a_{m-3}x_2 + \cdots + a_0x_{m-1} = b_{m-1}
\end{gathered}$$

This system of equations is equivalent to the equation $\boldsymbol{a}\cdot\boldsymbol{x} = \boldsymbol{b}$. The function defined above is called a determinant of the element $\boldsymbol{a} \in \mathbf{C}_m(\boldsymbol{\varepsilon})$. It may be proved (see [3, 4]) that the following identity holds for every element $\boldsymbol{x} \in \mathbf{C}_m(\boldsymbol{\varepsilon})$:

$$\begin{aligned}
\Delta(\boldsymbol{x}) = \begin{vmatrix} x_0 & x_{m-1} & \ldots & x_1\\ x_1 & x_0 & \ldots & x_2\\ \ldots & \ldots & \ldots & \ldots\\ x_{m-1} & x_{m-2} & \ldots & x_0 \end{vmatrix} &= \boldsymbol{x}\cdot\boldsymbol{x}(\alpha_m)\cdot\boldsymbol{x}(\alpha_m^2)\cdot\ldots\cdot\boldsymbol{x}(\alpha_m^{m-1})\\
&= (\operatorname{sp}\boldsymbol{x})(\operatorname{sp}\boldsymbol{x}(\alpha_m))(\operatorname{sp}\boldsymbol{x}(\alpha_m^2))\ldots(\operatorname{sp}\boldsymbol{x}(\alpha_m^{m-1})),
\end{aligned} \tag{2}$$

where

$$\operatorname{sp}\boldsymbol{x} \equiv \operatorname{sp}(x_0 + x_1\boldsymbol{\varepsilon} + x_2\boldsymbol{\varepsilon}^2 + \ldots + x_{m-1}\boldsymbol{\varepsilon}^{m-1}) = x_0 + x_1 + x_2 + \cdots + x_{m-1}.$$

For the example, if $\boldsymbol{x} = x_0 + x_1\boldsymbol{\varepsilon} + x_2\boldsymbol{\varepsilon}^2 \in \mathbf{C}_3(\boldsymbol{\varepsilon})$, we obtain:

$$\Delta(\boldsymbol{x}) = \begin{vmatrix} x_0 & x_2 & x_1\\ x_1 & x_0 & x_2\\ x_2 & x_2 & x_0 \end{vmatrix} = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2$$

$$\begin{aligned}
&= (x_0 + x_1\boldsymbol{\varepsilon} + x_2\boldsymbol{\varepsilon}^2)\cdot(x_0 + \alpha_m x_1\boldsymbol{\varepsilon} + \alpha_m^2 x_2\boldsymbol{\varepsilon}^2)\cdot(x_0 + \alpha_m^2 x_1\boldsymbol{\varepsilon} + \alpha_m x_2\boldsymbol{\varepsilon}^2)\\
&= (x_0 + x_1 + x_2)\cdot(x_0 + \alpha_m x_1 + \alpha_m^2 x_2)\cdot(x_0 + \alpha_m^2 x_1 + \alpha_m x_2).
\end{aligned}$$

Using multiplicative property of the resolvent, we obtain that the first part of the identity (2) gives the multiplicative property of the determinant of elements of cyclic algebras. It means that for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}_m(\boldsymbol{\varepsilon})$ we have:

$$\Delta(\boldsymbol{x} \cdot \boldsymbol{y}) = \Delta(\boldsymbol{x})\Delta(\boldsymbol{y}) \tag{3}$$

By this way the determinant of an arbitrary element may represent a multiplicative function $F(\boldsymbol{x})$ which determines the composition property of cyclic algebras. However, the determinant is not unique multiplicative function of a cyclic algebra. Another multiplicative function is a norm of an arbitrary element of cyclic algebra which is defined by

$$N(\boldsymbol{x}) = \boldsymbol{x}(\alpha_m) \cdot \boldsymbol{x}(\alpha_m^2) \cdot \ldots \cdot \boldsymbol{x}(\alpha_m^{m-1}).$$

The multiplicativity of the norm follows from the multiplicativity of determinant $\Delta(\boldsymbol{x})$ and the trace $\operatorname{sp}\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbf{C}_m(\boldsymbol{\varepsilon})$. Therefore we have for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{C}_m(\boldsymbol{\varepsilon})$:

$$N(\boldsymbol{x} \cdot \boldsymbol{y}) = N(\boldsymbol{x})N(\boldsymbol{y}) \tag{4}$$

Multiplicative functions of the determinant and of the norm of any element are defined for every algebra $\mathbf{C}_m(\boldsymbol{\varepsilon})$. But for concrete cyclic algebra, others multiplicative functions may be found.

Denote by $\mathbf{E}_m \subset \mathbf{C}_m(\boldsymbol{\varepsilon})$ a linear space generated by elements $\boldsymbol{\varepsilon}^k$. Let us remark that for any element $\boldsymbol{t} = t_1\boldsymbol{\varepsilon} + t_2\boldsymbol{\varepsilon}^2 + \ldots + t_{m-1}\boldsymbol{\varepsilon}^{m-1} \in \mathbf{E}_m$ we have the following identity (see [3]):

$$\Delta(\exp \boldsymbol{t}) = \prod_{k=1}^{m-1} \Delta(\exp t_k\boldsymbol{\varepsilon}^k) = 1. \tag{5}$$

The above provisions were obtained for complex cyclic algebras. However, identities (2), (3), (4) and (5) remain valid for arbitrary elements of real cyclic algebras and it is obvious that for $\boldsymbol{x} \in \mathbf{R}_m^{\pm}(\boldsymbol{\varepsilon})$ we have $\Delta(\boldsymbol{x}) \in \mathbf{R}$ and $N(\boldsymbol{x}) \in \mathbf{R}$. By this way, we may on linear spaces of algebras $\mathbf{R}_m^{\pm}(\boldsymbol{\varepsilon})$ construct geometric structures the fundamental forms $\boldsymbol{g}_h(\boldsymbol{x})$ of which are multiplicative functions above or other ones. Groups of motions of such spaces are subgroups of the group of linear transformations of a following type:

$$\boldsymbol{x}' = (\exp \boldsymbol{t}) \cdot \boldsymbol{x}, \quad \boldsymbol{t} \in \mathbf{V}_m \subset \mathbf{E}_m \tag{6}$$

where $\mathbf{V}_m$ is some subspace of linear space $\mathbf{E}_m$.

Especially, putting $\boldsymbol{g}_m(\boldsymbol{x}) = \Delta(\boldsymbol{x})$ and $\mathbf{V}_m \equiv \mathbf{E}_m$ we obtain a geometric structure on whole linear space of an arbitrary algebra $\mathbf{R}_m^{\pm}(\boldsymbol{\varepsilon})$. Geometry of such spaces is called cyclic geometry. Vector length of $\boldsymbol{x} \in \mathbf{R}_m^{\pm}(\boldsymbol{\varepsilon})$ in these geometries is defined by $[\boldsymbol{x}] = \sqrt[m]{\Delta(\boldsymbol{x})}$ and its value may be real as well as complex. Thus as a extension of a vector the modul $|[\boldsymbol{x}]|$ of its length may be used. A vector with zero length will be called isotropic.

A set of vectors with constant length ϱ forms a surface which is called a cyclic spheroid with radius ϱ. This spheroid is in a center affine space of cyclic or anticyclic algebra determined by an equation $\Delta(\boldsymbol{x}) = 0$. If $\varrho = 0$, then the spheroid is called isotropic. It divides all centre affine space of given algebra into disjunctive parts which are called quadrants of a cyclic space.

In every such quadrant, there may be a cyclic angle between vectors of the same length $\boldsymbol{x}$ and $\boldsymbol{x}' = (\exp \boldsymbol{t}) \cdot \boldsymbol{x}$ of the given quadrant defined as an element $\boldsymbol{t} \in \mathbf{E}_m$. If vectors do not have the same length, their length must be normalised first.

A model example of the cyclic geometry may be obtained due to a two-dimensional pseudo Euclidean space, because for $m = 2$ we have

$$\boldsymbol{g}_2(\boldsymbol{x}) = \Delta(\boldsymbol{x}) = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_0 \end{vmatrix} = x_0^2 - x_1^2$$

for any $\boldsymbol{x} = x_0 + x_1\boldsymbol{\varepsilon} \in \mathbf{R}_2^+(\boldsymbol{\varepsilon})$. In this case, spheroids with a real radius $x_0^2 - x_1^2 = \varrho^2$ are represented by hyperbolas with equations $x_0^2 - x_1^2 = \varrho^2$, spheroids with a purely imaginary radius $\mathrm{i}\varrho$ are hyperbolas with equations $x_0^2 - x_1^2 = \varrho^2$ and an isotropic spheroid is a couple of intersecting lines with equations $x_0 = \pm x_1$.

Further, a cyclic angle between vectors $\boldsymbol{x} = x_0 + x_1\boldsymbol{\varepsilon}$ and $\boldsymbol{x}' = (\exp \beta \boldsymbol{t}) \cdot \boldsymbol{x}$ will be equal to $\beta\boldsymbol{\varepsilon}$. It differs from the pseudo euclidean angle β only by a multiple $\boldsymbol{\varepsilon}$.

The second model example is represented by a cyclic space of an algebra $\mathbf{R}_3^+(\boldsymbol{\varepsilon})$. The fundamental form will be given as

$$\boldsymbol{g}_2(\boldsymbol{x}) = \Delta(\boldsymbol{x}) = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2,$$

so that spheroids of a positive radius $\varrho > 0$ will be surfaces with equation $x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = \varrho^3$, spheroids with a negative radius $-\varrho < 0$ will be surfaces with equation $x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = -\varrho^3$ and an isotropic spheroid will be an plane with equation $x_0 + x_1 + x_2 = 0$ and a line $x_0 = x_1 = x_2$. To imagine the spheroid in 3-dimmesional center affine space we study the intersections of this spheroid and coordinate planes with equations $x_0^3 + x_1^3 - 3x_0x_1c = \varrho^3 - c^3$ (for $c = \varrho$ we get a "Descartes leaf"—a curve with equation $x_0^3 + x_1^3 - 3\varrho x_0x_1$). However, transforming a coordinate system by relations

$$x = \frac{x_0 + x_1 + x_2}{\sqrt{3}}, \quad y = \frac{x_0 - x_1}{\sqrt{2}}, \quad z = \frac{x_0 + x_1 - 2x_2}{\sqrt{6}}$$

and

$$x_0 = \frac{x}{\sqrt{3}} + \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{6}}, \quad x_1 = \frac{x}{\sqrt{3}} - \frac{y}{\sqrt{2}} + \frac{z}{\sqrt{6}}, \quad x_2 = \frac{x}{\sqrt{3}} - \frac{2z}{\sqrt{6}}$$

we may the equation of spheroid with a radius $\varrho > 0$ rearrange into form

$$x = 2\varrho^3/3\sqrt{3}(y^2 + z^2),$$

since the axis Ox is identical with a line $x_0 = x_1 = x_2$ and axes Oy and Oz lie in a plane $x_0 + x_1 + x_2 = 0$.

It follows from this that a surface showing a central spheroid of cyclic geometry of the third order may be obtained by a rotation of quadratic hyperbole with equation

$$x = \frac{2\varrho^3}{3\sqrt{3}y^2}$$

around the axis Ox.

A cyclic angle between vectors $\boldsymbol{x}$ and $\boldsymbol{x}' = \exp(t_1\boldsymbol{\varepsilon} + t_2\boldsymbol{\varepsilon}^2) \cdot \boldsymbol{x}$ in $\mathbf{R}_3^+(\boldsymbol{\varepsilon})$ is two parametric and it is equal to a vector $\boldsymbol{t} = t_1\boldsymbol{\varepsilon} + t_2\boldsymbol{\varepsilon}^2 \in \mathbf{T}_3$. However, in $\mathbf{R}_3^+(\boldsymbol{\varepsilon})$ there may be defined a scalar angle between vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$ as a trace of the element $\boldsymbol{t}$.

Let us remark, that Euclidean planimetry is a model example of anticyclic geometry because $\mathbf{R}_2^-(\boldsymbol{\varepsilon}) \equiv \mathbf{C}$.

Beside of cyclic and anticyclic spaces defined on the all linear space of algebras $\mathbf{R}_m^\pm(\boldsymbol{\varepsilon})$, some geometric structure may be introduced on a hyperplane $\mathbf{\Pi}_{m-1}\colon x_0 + x_1 + \ldots + x_{m-1} = 1$ if a norm of $\boldsymbol{x} \in \mathbf{\Pi}_{m-1} \subset \mathbf{R}_m^\pm(\boldsymbol{\varepsilon})$ is taken as a fundamental form and transformations given by functions $\boldsymbol{x}' = \boldsymbol{x} \cdot \exp(t_1\boldsymbol{\varepsilon} + t_2\boldsymbol{\varepsilon}^2 + \ldots + t_{m-1}\boldsymbol{\varepsilon}^{m-1})$ with $t_1 + t_2 + \ldots + t_{m-1} = 0$ are taken as motions because these transformations put $\mathbf{\Pi}_{m-1}$ anew onto $\mathbf{\Pi}_{m-1}$ and preserve a norm. The geometry acquired by this way is called a projective cyclic geometry because components of an element $\boldsymbol{x} \in \mathbf{\Pi}_{m-1} \subset \mathbf{R}_m^\pm(\boldsymbol{\varepsilon})$ represent barycentric coordinates of points of hyperplane $\mathbf{\Pi}_{m-1} \subset \mathbf{R}_m^\pm(\boldsymbol{\varepsilon})$.

There exists another series of spaces the geometry of which is connected with cyclic and anticyclic algebras. It arises on spheroids in cyclic and anticyclic spaces of algebras $\mathbf{R}_m^\pm(\boldsymbol{\varepsilon})$, $m > 2$. Motions of spheroids are given by linear functions (6). It is immediately seen if we take an element $\exp\boldsymbol{t} = \exp(t_1\boldsymbol{\varepsilon} + t_2\boldsymbol{\varepsilon}^2 + \ldots + t_{m-1}\boldsymbol{\varepsilon}^{m-1})$ as a parametric equations of a spheroid and use a known property of exponential function $\exp\boldsymbol{a} \cdot \exp\boldsymbol{b} = \exp(\boldsymbol{a} + \boldsymbol{b})$. For example, for $m = 3$ parametric equations have a form $x_0 = \varrho A(t_1, t_2)$, $x_1 = \varrho B(t_1, t_2)$, $x_2 = \varrho C(t_1, t_2)$, where we put $\exp(t_1\boldsymbol{\varepsilon} + t_2\boldsymbol{\varepsilon}^2) = A(t_1, t_2) + B(t_1, t_2)\boldsymbol{\varepsilon} + B(t_1, t_2)\boldsymbol{\varepsilon}^2$.

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