Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **55**, 1 (2016) 47–52

On Uniqueness Theorems for Ricci Tensor*

Marina B. KHRIPUNOVA [1a], Sergey E. STEPANOV [1b],
Irina I. TSYGANOK [1c], Josef MIKEŠ [2]

[1] *Department of Mathematics, Finance University under the Government of Russian Federation, Leningradsky Prospect, 49-55, 125468 Moscow, Russian Federation*
[a] *e-mail: MBKHripunova@fa.ru*
[b] *e-mail: s.e.stepanov@mail.ru*
[c] *e-mail: i.i.tsyganok@mail.ru*

[2] *Department of Algebra and Geometry, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic*
e-mail: josef.mikes@upol.cz

(Received October 25, 2015)

Abstract

In Riemannian geometry the prescribed Ricci curvature problem is as follows: given a smooth manifold M and a symmetric 2-tensor r, construct a metric on M whose Ricci tensor equals r. In particular, DeTurck and Koiso proved the following celebrated result: the Ricci curvature uniquely determines the Levi-Civita connection on any compact Einstein manifold with non-negative section curvature. In the present paper we generalize the result of DeTurck and Koiso for a Riemannian manifold with non-negative section curvature. In addition, we extended our result to complete non-compact Riemannian manifolds with nonnegative sectional curvature and with finite total scalar curvature.

Key words: Uniqueness theorem for Ricci tensor, compact and complete Riemannian manifolds, vanishing theorem.

2010 Mathematics Subject Classification: 53C20

1 Introduction

The main point of the papers [1, 2] and the monograph [3, pp. 140–153] is that in certain circumstances the metric (or at last the connection) is uniquely

*Supported by the project IGA PrF 2014016 Palacky University Olomouc.

determined by the Ricci tensor. In particular, in [1, Corollary 3.3] and [3, Theorem 5.42] anyone can read the following: Let $(M, \bar{g})$ be a compact *Einstein manifold* with non-negative section curvature and with the Ricci tensor $\mathrm{Ric}(\bar{g}) = \bar{g}$, then another Riemannian metric g on M with $\mathrm{Ric}(g) = \bar{g}$ has the same Levi-Civita connection as $\bar{g}$. We note that this proposition is a corollary of the Eells and Sampson *vanishing theorem* for harmonic maps of compact Riemannian manifolds (see [4, p. 124]).

In the present paper we consider a compact *Riemannian manifold* $(M, \bar{g})$ with non-negative sectional curvature and with $\mathrm{Ric}(\bar{g}) \leq \bar{g}$. Under these conditions, we prove that if g is another Riemannian metric on M with the Ricci tensor $\mathrm{Ric}(g) = \bar{g}$, then g and $\bar{g}$ have the same Levi-Civita connection. Furthermore, if the full holonomy group $\mathrm{Hol}(\bar{g})$ is irreducible then the metric $g = C\bar{g}$ for some constant > 0. In turn, it is well known that $\mathrm{Ric}(\bar{g}) = \mathrm{Ric}(C\bar{g})$. This proposition was announced in report [5] at the 12th International Conference on Geometry and Applications (September 1–5, 2015, Varna, Bulgaria). We extend the above scheme to show that if $(M, \bar{g})$ is a non-compact manifold $(M, \bar{g})$ with non-negative sectional curvature and with the Ricci tensor $\mathrm{Ric}(\bar{g}) \leq \bar{g}$ then there is no complete Riemannian metric g such that its Ricci tensor $\mathrm{Ric}(g) = \bar{g}$ and its total scalar curvature $s_g(M)$ is finite. This proposition is a corollary of the Schoen and Yau *vanishing theorem* for harmonic maps of complete non-compact Riemannian manifolds (see [8]).

Our statements generalize and complement the results of the papers [1] and [2], and the monograph [3].

2 Harmonic maps

For the discussion of harmonic maps we will follow Eells and Sampson [4]. Let (M, g) and $(\bar{M}, \bar{g})$ be two Riemannian manifolds with the Levi-Civita connections $\nabla := \nabla(g)$ and $\bar{\nabla} := \nabla(\bar{g})$, and $f\colon (M, g) \to (\bar{M}, \bar{g})$ be a smooth map. The *energy density* of f is defined as the scalar function

$$e(f) = 2^{-1} \|df\|^2 \tag{1}$$

where $\|df\|^2$ is the squared norm of the differential of f with respect to metric on the bundle $T^*M \otimes f^*T\bar{M}$. Then the *total energy* of f is obtained by integrating the energy density $e(f)$ over M

$$E(f) = \int_M e(f)\, \mathrm{d}Vol_g \tag{2}$$

where $\mathrm{d}v_g$ denotes the measure on (M, g) induced by the metric g. If f is of class C^2 and $E(f) < +\infty$, and f is an extremum of the *Dirichlet energy functional* $E(f)$, then f is called a *harmonic map* and satisfies the *Euler–Lagrange equation*

$$\mathrm{trace}_g\, D\, \mathrm{d}f = 0 \tag{3}$$

where D is the connection in the bundle $T^*M \otimes f^*T\bar{M}$ induced from the Levi-Civita connections ∇ and $\bar{\nabla}$ of (M, g) and $(\bar{M}, \bar{g})$, respectively.

For any harmonic $f\colon (M,g)\to(\bar{M},\bar{g})$ we have the *Weitzenböck formula* (see [4])

$$\Delta e(f) = Q(f) + \|D\,\mathrm{d}f\|^2 \tag{4}$$

where Δ is the Laplace–Beltrami operator $\Delta = \operatorname{div}\nabla$ and

$$Q(f) = g(\mathrm{Ric}, f^*\bar{g}) - \mathrm{trace}_g(\mathrm{trace}_g(f^*\overline{\mathrm{Riem}})) \tag{5}$$

where $\mathrm{Ric} = \mathrm{Ric}(g)$ is the Ricci tensor of (M,g) and $\overline{\mathrm{Riem}}$ is the Riemannian curvature tensor of $(\bar{M},\bar{g})$. Let the inequality $\overline{\sec}\le 0$ be satisfied anywhere on $(\bar{M},\bar{g})$ and the inequality $\mathrm{Ric}\ge 0$ be satisfied anywhere on compact (M,g), then $Q(f)$ is non-negative everywhere on M. Since our hypothesis implies that the left hand side of (3) is non-negative, then using the *Hopf's lemma* (see [6, pp. 30–31]), one can verify that $e(f)$ is constant. In this case, from (4) we obtain $D\,\mathrm{d}f = 0$. In this case, f is totally *geodesic map* (see [7]). Now we can formulate the following *vanishing theorem* on harmonic maps. Namely, if $f\colon (M,g)\to(\bar{M},\bar{g})$ is any harmonic mapping between a compact Riemannian manifold (M,g) with the Ricci tensor $\mathrm{Ric}\ge 0$ and a Riemannian manifold $(\bar{M},\bar{g})$ with the sectional curvature $\overline{\sec}\le 0$ then f is *totally geodesic* and has constant *energy density* $e(f)$. Furthermore, if there is at least one point of M at which its Ricci curvature $\mathrm{Ric}>0$, then every harmonic map $f\colon (M,g)\to(\bar{M},\bar{g})$ is constant (see [4, p. 124]).

In turn, Schoen and Yau have showed in [8] that $\sqrt{e(f)}$ is subharmonic function on (M,g) if $Q(f)\ge 0$. On other hand, Yau has proved in other his paper [9] that every non-negative L^2-integrable subharmonic function on a complete Riemannian manifold must be constant. Applying this to $\sqrt{e(f)}$, we conclude that $\sqrt{e(f)}$ is a constant if the total energy $E(f)<+\infty$ (see also [8]). On the other hand, every complete non-compact Riemannian manifold with nonnegative Ricci curvature has infinite volume (see [9]). In our case, we have $\mathrm{Ric}\ge 0$ then the volume of (M,g) is infinite. This forces the constant $e(f)$ to be zero and f to be a constant map (see also [8]). Now we can formulate another celebrated *vanishing theorem* on harmonic maps: If the sectional curvature of $(\bar{M},\bar{g})$ is non-positive and (M,g) is a complete non-compact manifold with $\mathrm{Ric}\ge 0$, then any harmonic map $f\colon (M,g)\to(\bar{M},\bar{g})$ with the finite energy $E(f)$ is a constant map (see [8], [10, p. 116]). We remark that in the original paper [8] the manifold $(\bar{M},\bar{g})$ was assumed to be compact. However, this assumption is superfluous (see [10, p. 116]).

3 The main theorem

If we consider the manifold M with two Riemannian metrics g and $\bar{g}$ then the identity mapping $\mathrm{Id}\colon (M,g)\to(M,\bar{g})$ is harmonic if and only if the deformation tensor $T=\bar{\nabla}-\nabla$ is a section of the tensor bundle $TM\otimes S_0^2M$, because in this case the Euler–Lagrange equation (3) has the form $\mathrm{trace}_g T = 0$ (see [1, 3]). In particular, if (M,g) is a manifold of strictly positive Ricci Ric curvature, then $\mathrm{Id}\colon (M,g)\to(M,\mathrm{Ric})$ is a harmonic map (see [1]). Next we can formulate and prove the following

Theorem 1 *Let $(M, \bar{g})$ be a compact Riemannian manifold with the sectional curvature $\overline{\sec} \geq 0$ and with the Ricci tensor $\overline{\mathrm{Ric}} \leq \bar{g}$. If g is another Riemannian metric on M with the Ricci tensor $\mathrm{Ric} = \bar{g}$, then g and $\bar{g}$ have the same Levi-Civita connection. Furthermore, if the full holonomy group $\mathrm{Hol}(\bar{g})$ of $(M, \bar{g})$ is irreducible then $\mathrm{Ric} = \overline{\mathrm{Ric}}$.*

Proof With the above assumptions, we have $\mathrm{Ric} = \bar{g} > 0$, then the identity map $\mathrm{Id}\colon (M, g) \to (M, \bar{g})$ is harmonic. In this case, we have $e(f) = \frac{1}{2}s$ for the energy density $e(f)$ of the harmonic identity map $\mathrm{Id}\colon (M, g) \to (M, \bar{g})$ and the scalar curvature $s = \mathrm{trace}_g \mathrm{Ric}$ of the Riemannian manifold (M, g) (see [1], [3, p. 152]). Therefore, s satisfies the Weitzenbck formula (4) which has the following form (see [1]):

$$\frac{1}{2}\Delta s = Q(f) + \|D\,\mathrm{d}f\|^2 \tag{6}$$

where $Q(f) = g^{ik}g^{jl}(\bar{g}_{ij}\bar{g}_{kl} - \bar{R}_{ijkl}$ and $\|D\,\mathrm{d}f\|^2 = g^{ij}g^{kl}\bar{g}_{pq}T^p_{ik}T^q_{jl} \geq 0$ for local components g_{ij}, $\bar{g}_{kl}$, $\bar{R}_{ijkl}$ and T^i_{kl} of metric tensors g and $\bar{g}$, the Riemannian curvature tensor $\overline{\mathrm{Riem}}$ and the deformation tensor T, respectively. On the other hand, we have the identity (see [3, p. 436], [11])

$$(\bar{g}_{ij}\bar{R}_{kl} - \bar{R}_{ijkl})\varphi^{ik}\varphi^{jl} = \sum_{i<j}\overline{\sec}(\bar{e}_i, \bar{e}_j)(\bar{\lambda}_i, \bar{\lambda}_j)^2 \tag{7}$$

where φ is any smooth symmetric tensor field such that $\varphi(\bar{e}_i, \bar{e}_j) = \bar{\lambda}_i\delta_{ij}$ for the Kronecker delta δ_{ij} and some orthonormal basis $\{\bar{e}_1, \dots, \bar{e}_n\}$ at any point $x \in M$. Then equation (6) can be rewritten in the form

$$\frac{1}{2}\Delta s = \sum_{i<j}\overline{\sec}(\bar{e}_i, \bar{e}_j)(\bar{\lambda}_i, \bar{\lambda}_j)^2 + g^{ik}g^{jl}\bar{g}_{ij}(\bar{g}_{kl} - \bar{R}_{kl}) + \|T\|^2 \tag{8}$$

where $g(\bar{e}_i, \bar{e}_j) = \bar{\lambda}_i\delta_{ij}$. We remark that under the stated assumptions the right side of (8) is non-negative, since then $\Delta s \geq 0$. Therefore, the scalar curvature s is a positive subharmonic function on (M, g). If $(M, \bar{g})$ is a compact Riemannian manifold, then using the Hopf's lemma (see [6, pp. 30–31]), one can verify that $s = \mathrm{const}$. In this case, from (8) we obtain $T = 0$. Then g and $\bar{g}$ have the same Levi-Civita connection, i.e. $\bar{\nabla} g = 0$. Furthermore, if the full holonomy group $\mathrm{Hol}(\bar{g})$ of $(M, \bar{g})$ is irreducible then the metric $g = C\bar{g}$ for some constant > 0 (see [3, pp. 282, 285–287]). In this case, we have the identity $\mathrm{Ric} = \overline{\mathrm{Ric}}$ because $\mathrm{Ric}(\bar{g}) = \mathrm{Ric}(C\bar{g})$ for some positive constant C (see [3, pp. 44, 152]). □

4 Two vanishing theorems

In [13] the following non-existence theorem was proved: Let $(M, \bar{g})$ be a compact Riemannian manifold with all sectional curvature less then $(\bar{n} - 1)^{-1}$. Then there is no Riemannian metric g on M such that its Ricci tensor $\mathrm{Ric} = \bar{g}$. In its turn, in [2] the following vanishing theorem was proved: Let $\bar{g}$ be a metric on a compact manifold M with the sectional curvature $\overline{\sec} < +1$, then any metric g does not exist on M such that its Ricci tensor $\mathrm{Ric} = \bar{g}$. We also get a non-existence result which complements the above propositions. In turn, we can formulate and prove an analogue of these propositions in the following form.

Theorem 2 *Let $(M, \bar{g})$ be a compact Riemannian manifold with nonnegative section curvatures and with the Ricci tensor $\overline{\mathrm{Ric}} \leq \bar{g}$. If in addition there is at least one point of M at which the Ricci tensor $\overline{\mathrm{Ric}} < \bar{g}$, then there is no Riemannian metric g on M such that its Ricci tensor $\mathrm{Ric} = \bar{g}$.*

Proof Let M be a compact manifold. We may assume that M is oriented by taking the twofold covering of M if necessary. Then by Green's theorem (see [6, pp. 31–33]) we obtain from (6) the following identity

$$\int_M Q(f)\,\mathrm{d}Vol_g + \int_M \|T\|^2\,\mathrm{d}Vol_g = 0. \tag{9}$$

If the inequalities $\overline{\sec} \geq 0$ and $\overline{\mathrm{Ric}} \leq \bar{g}$ are satisfied and there is a one point x of M in which $\overline{\mathrm{Ric}} < \bar{g}$ then the inequality $\int_M Q(f)\,\mathrm{d}Vol_g > 0$ holds. This inequality contradicts the equation (9). In this case, the harmonic mapping f must be constant. □

Theorem 3 *Let $(M, \bar{g})$ be a non-compact Riemannian manifold with the section curvature $\overline{\sec} \geq 0$ and with the Ricci tensor $\overline{\mathrm{Ric}} < \bar{g}$. Then there is no complete Riemannian metric g on $(M, \bar{g})$ such that its Ricci tensor $\mathrm{Ric} = \bar{g}$ and its total scalar curvature $s(M)$ is finite.*

Proof Let $(M, \bar{g})$ be a non-compact Riemannian manifold with the section curvature $\overline{\sec} \geq 0$ and with the Ricci tensor $\overline{\mathrm{Ric}} < \bar{g}$, then $Q(f)$ is non-negative everywhere on M. If we assume that there is complete Riemannian metric g on $(M, \bar{g})$ such that its Ricci tensor $\mathrm{Ric} = \bar{g} > 0$, then the volume of (M, g) is infinite (see [9]). Moreover, we have $e(f) = \frac{1}{2}s$ for the energy density $e(f)$ of the harmonic identity map $\mathrm{Id}\colon (M, g) \to (M, \bar{g})$ and the scalar curvature $s = \mathrm{trace}_g \mathrm{Ric}$ of the Riemannian manifold (M, g). In this case, $\sqrt{s}$ is a strictly positive subharmonic function on a complete Riemannian manifold (M, g) of infinite volume (see [8]). In addition, if we suppose that the total scalar curvature $\int_M s\,\mathrm{d}Vol_g < +\infty$, then s must be zero (see [8], [12, p. 262]). On the other hand, according to the condition of our theorem the scalar curvature $s = \mathrm{trace}_g\,\bar{g} > 0$ and hence there is no complete Riemannian metric g on non-compact $(M, \bar{g})$ such that its Ricci tensor $\mathrm{Ric} = \bar{g}$. □

References

[1] DeTurck, D., Koiso, N.: *Uniqueness and non-existence of metrics with prescribed Ricci curvature*. Annales de l'Institut Henri Poincare (C) Analyse non lineaire **1**, 5 (1984), 351–359.

[2] Hamilton, R. S.: The Ricci curvature equation. Lecture notes: Seminar on nonlinear partial differential equations, *Mathematical Sciences Research Institute Publications*, Berkeley, 1983, 47–72.

[3] Becce, A. L.: Einstein manifolds. *Springer-Verlag*, Berlin–Heidelberg, 1987.

[4] Eells, J., Sampson, J. H.: *Harmonic mappings of Riemannian manifolds.* American Journal of Mathematics **86**, 1 (1964), 109–160.

[5] Stepanov, S., Tsyganok, I.: *Vanishing theorems for projective and harmonic mappings.* Journal of Geometry **106**, 3 (2015), 640–641.

[6] Yano, K., Bochner, S.: Curvature and Betti numbers. *Princeton Univ. Press*, Princeton, 1953.

[7] Vilms, J.: *Totally geodesic maps.* Journal of Differential Geometry **4**, 1 (1970), 73–79.

[8] Schoen, R., Yau, S. T.: *Harmonic maps and topology of stable hypersurfaces and manifolds with non-negative Ricci curvature.* Commenttarii Mathematici Helvetici **51**, 1 (1976), 333–341.

[9] Yau, S. T.: *Some function-theoretic properties of complete Riemannian manifold and their applications to geometry.* Indiana University Mathematics Journal **25**, 7 (1976), 659–679.

[10] Yau, S. T. (eds.): Seminar on Differential Geometry. Annals of Mathematics Studies, **102**, *Princeton Univ. Press*, Princeton, NJ, 1982.

[11] Berger, M., Ebin, D.: *Some decompositions of the space of symmetric tensors on a Riemannian manifold.* Journal of Differential Geometry **3**, 3-4 (1969), 379–392.

[12] Pigola, S., Rigoli, M., Setti, A. G.: Vanishing and finiteness results in geometric analysis. A generalization of the Bochner technique. *Birkhäuser*, Basel, 2008.