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The Killing Tensors on an n-dimensional Manifold with $SL(n, \mathbb{R})$-structure

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Abstract

In this paper we solve the problem of finding integrals of equations determining the Killing tensors on an n-dimensional differentiable manifold M endowed with an equiaffine $SL(n, \mathbb{R})$-structure and discuss possible applications of obtained results in Riemannian geometry.

Key words: Differentiable manifold, $SL(n, \mathbb{R})$-structure, Killing tensors.

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1 Introduction

1.1. The "structural point of view" of affine differential geometry was introduced by K. Nomizu in 1982 in a lecture at Münster University with the title "What is Affine Differential Geometry?" (see [12]). In the opinion of K. Nomizu, the geometry of a manifold M endowed with an *equiaffine structure* is called affine differential geometry.

In recent years, there has been a new ware of papers devoted to affine differential geometry. Today the number of publications (including monographs) on affine differential geometry reached a considerable level. The main part of these publications is devoted to geometry of hypersurfaces (see [15, 16] for the history and references).

1.2. In the present paper we solve the problem of finding integrals of equations determining the Killing tensors (see [8] for the definitions, properties and applications) on an n-dimensional differentiable manifold M endowed with an equiaffine structure. The paper is a direct continuation of [18]. The same notations are used here.

The first of two present theorems proved in our paper is an affine analog of the statement published in the paper [17], which appeared in the process of solving problems in General relativity.

2 Definitions and results

2.1. In order to clarify the approach to problem of finding integrals of equations determining the Killing tensors on an n-dimensional differentiable manifold M we shall start with a brief introduction to the subject which emphasizes the notion of an equiaffine $SL(n,\mathbb{R})$-structure.

Let M be a connected differentiable manifold of dimension n $(n > 2)$, and let $L(M)$ be the corresponding bundle of linear frames with structural group $GL(n,\mathbb{R})$. We define $SL(n,\mathbb{R})$-structure on M as a principal $SL(n,\mathbb{R})$- subbundle of $L(M)$. It is well known that an $SL(n,\mathbb{R})$-structure is simply a volume element on M, i.e. an n-form η that does not vanishing anywhere (see [6, Chapter I, §2]).

We recall the famous problem of the existence of a uniquely determined linear connection ∇ reducible to G for each given G-structure on M (see [1, p. 213]). For example, if M is a manifold with a pseudo-Riemannian metric g of an arbitrary index k, then the bundle $L(M)$ admits a unique linear connection ∇ without torsion that is reducible to $O(m,k)$-structure. Such a connection is called the *Levi-Civita connection.* It is characterized by the following condition $\nabla g = 0$.

A linear connection ∇ having zero torsion and reducible to $SL(n,\mathbb{R})$ is said to be *equiaffine* and can be characterized by the following equivalent conditions (see [15, p. 99], [16, pp. 57–58]):

(1) $\nabla\eta = 0$;

(2) the Ricci tensor Ric of ∇ is symmetric; that means $\mathrm{Ric}(X,Y) = \mathrm{Ric}(Y,X)$ for any vector fields $X, Y \in C^\infty TM$.

An *equiaffine* $SL(n,\mathbb{R})$*-structure* or an *equiaffine structure* on an n-dimensional differentiable manifold M is a pair (η,∇), where ∇ is a linear connection with zero torsion and η is a volume element which is parallel relative to ∇ (see [13, p. 43]).

The curvature tensor R of an equiaffine connection ∇ admits a point-wise $SL(n,\mathbb{R})$-invariant decomposition of the form

$$R = (n-1)^{-1}[\mathrm{id}_M \otimes \mathrm{Ric}\text{-}\mathrm{Ric} \otimes \mathrm{id}_M] + W$$

where W is the *Weyl projective curvature tensor* (see [16, p. 73–74], [2, §40]). Then two classes of equiaffine structures can be distinguished in accordance with this decomposition: the *Ricci-flat* equiaffine $SL(n,\mathbb{R})$-structures for which $\mathrm{Ric}=0$, and the *equiprojective* $SL(n,\mathbb{R})$-structures for which

$$R=(n-1)^{-1}[\mathrm{id}_M\otimes \mathrm{Ric}\text{-}\mathrm{Ric}\otimes \mathrm{id}_M].$$

Remark 1 Recall that a linear connection ∇ with zero torsion is called *Ricci-flat* if the Ricci tensor $\mathrm{Ric}=0$ (see [9]). On the anther hand, a connection ∇ is called *equiprojective* if the Weyl projective curvature tensor $W=0$ (see [15, §18]). In the literature equiprojective connections sometimes are called *projectively flat* (see, for example, [16, p. 73]).

An autodiffeomorphism of the manifold M is an automorphism of $SL(n,\mathbb{R})$-structure if and only if it preserves the volume element η. Let X be a vector field on M. The function $\mathrm{div}\,X$ defined by the formula $(\mathrm{div}\,X)\eta=L_X\eta$ where L_X is the Lie differentiation in the direction of the vector field X is called the divergence of X with respect to the n- form η (see [7, Appendix no. 6]). Obviously, X is an infinitesimal automorphism of an $SL(n,\mathbb{R})$-structure if and only if $\mathrm{div}\,X=0$. Such a vector field X is said to be *solenoidal*.

For an arbitrary vector field X on M with a linear connection ∇ we can introduce the tensor field $A_X=L_X-\nabla_X$ regarded as a field of linear endomorphisms of the tangent bundle TM. If M is an n-dimensional with an equiaffine $SL(n,\mathbb{R})$-structure then the formula $\mathrm{trace}\,A_X=-\,\mathrm{div}\,X$ can be verified directly (see [7, Appendix no. 6]).

We have the $SL(n,\mathbb{R})$-invariant decomposition

$$A_X=-n^{-1}(\mathrm{div}\,X)\,\mathrm{id}_M+\dot{A}_X$$

at every point $x\in M$.

Two classes of vector fields on M endowed with an equiaffine $SL(n,\mathbb{R})$-structure can be distinguished in accordance with this decomposition: the *solenoidal vector fields* and the *concircular vector fields* for which, by definition (see [14, p. 322], [9]), we have $A_X=-n^{-1}(\mathrm{div}\,X)\,\mathrm{id}_M$.

The integrability conditions of the structure equation $A_X=-n^{-1}(\mathrm{div}\,X)\,\mathrm{id}_M$ of the concircular vector field X is the Ricci's identity

$$Y(\mathrm{div}\,X)Z-Z(\mathrm{div}\,X)Y=nR(Y,Z)X$$

for any vector fields $Y,Z\in C^\infty TM$ (see [2, §11]). This identity are equivalent to the condition $W(Y,Z)X=0$ for any vector fields $Y,Z\in C^\infty TM$. It follows that an equiaffine $SL(n,\mathbb{R})$- structure on an n-dimensional manifold M is equiprojective if and only if there exist n linearly independent concircular vector fields $X_1,X_2,\ldots,X_p$ on M (see also [24]). This statement is an affine analog of the well known fact for the Riemannian manifold M of constant sectional curvature (see [3]).

Remark 2 A pseudo-Riemannian manifold (M,g) with a projectively flat Levi-Civita connection ∇ is a manifold of constant section curvature (see [15, §18]). Therefore a manifold M endowed with an equiprojective $SL(n,\mathbb{R})$-structure is an affine analog of a pseudo-Riemannian manifold of constant sectional curvature.

2.2. We consider an n-dimensional manifold M with an equiaffine $SL(n, \mathbb{R})$-structure and denote by $\Lambda^p M$ $(1 \leq p \leq n-1)$ the p^{th} exterior power $\Lambda^p(T^*M)$ of the cotangent bundle T^*M of M. Hence $C^\infty \Lambda^p M$, the space of all C^∞-sections of $\Lambda^p M$, is the space of skew-symmetric covariant tensor fields of degree p $(1 \leq p \leq n-1)$.

Let $\gamma\colon J \subset \mathbb{R} \to M$ be an arbitrary geodesic on M with affine parameter $t \in J$. In this case, we have $\nabla_{\frac{d\gamma}{dt}} \frac{d\gamma}{dt} = 0$ for the tangent vector $\frac{d\gamma}{dt}$ of γ.

Definition 1 (see [18]). A skew-symmetric tensor field $\omega \in C^\infty \Lambda^p M$ $(1 \leq p \leq n-1)$ on an n-dimensional manifold M with an equiaffine $SL(n, \mathbb{R})$-structure is called Killing-Yano tensor of degree p if the tensor

$$i_{\frac{d\gamma}{dt}} \omega := \operatorname{trace}\left(\frac{d\gamma}{dt} \otimes \omega\right)$$

is parallel along an arbitrary geodesic γ on M.

From this definition we conclude that

$$\left(\nabla_{\frac{d\gamma}{dt}} \omega\right)\left(\frac{d\gamma}{dt}, X_2, \ldots, X_p\right) = 0$$

for any vector fields $X_2, \ldots, X_p \in C^\infty TM$. Since the geodesic γ may be chosen arbitrary, the above relation is possible if and only if $\nabla\omega \in C^\infty \Lambda^{p+1} M$, which is equivalent to $d\omega = (n+1)\nabla\omega$ for the exterior differential operator $d\colon C^\infty \Lambda^p M \to C^\infty \Lambda^{p+1} M$.

Obviously, the set of Killing-Yano tensors of degree p $(1 \leq p \leq n-1)$ constitutes an $\mathbb{R}$-module of tensor fields on M, denoted by $\mathbf{K}^p(M, \mathbb{R})$.

Let $X_1, \ldots, X_p$ be p linearly independent concircular vector fields on M $(1 \leq p \leq n-1)$. Then direct inspection shows that the tensor field ω of degree $n-p$ dual to the tensor field $alt\{X_1 \otimes \cdots \otimes X_p\}$ relative to the n-form η is a Killing-Yano tensor (see also [18]). Therefore on any n-manifold M with equiprojective $SL(n, \mathbb{R})$-structure, there exist at least $n![p!(n-p)!]^{-1}$ linearly independent Killing-Yano tensors (see [18]). Moreover the following theorem is true.

Theorem 1 *On an n-dimensional manifold M endowed with an equiprojective $SL(n, \mathbb{R})$-structure (η, ∇), there exist a local coordinate system $x^1, \ldots, x^n$ in which an arbitrary Killing-Yano tensor ω of degree p $(1 \leq p \leq n-1)$ has the components*

$$\omega_{i_1 \ldots i_p} = e^{(p+1)\psi}(A_{i_0 i_1 \ldots i_p} x^{i_0} + B_{i_1 \ldots i_p}) \qquad (2.1)$$

where $A_{i_0 i_1 \ldots i_p}$ and $B_{i_1 \ldots i_p}$ are arbitrary constants skew-symmetric w.r.t. all their indices and $\psi = (n+1)^{-1} \ln(\eta)$.

From the theorem we conclude that the maximum of linearly independent the Killing–Yano tensors is by calculating the number K_n^p of independent $A_{i_0 i_1 \ldots i_p}$ and $B_{i_1 \ldots i_p}$ which exist after accounting for the symmetries on the indices. It follows that $K_n^p = \frac{(n+1)!}{(p+1)!(n-p)!}$ is the maximum number linearly independent the Killing–Yano tensors.

Corollary 1 *Let M be an n-dimensional manifold endowed with an equiprojective $SL(n,\mathbb{R})$-structure then*

$$\dim K^p(M,\mathbb{R}) = \frac{(n+1)!}{(p+1)!(n-p)!}.$$

On our fixed manifold M with an equiaffine $SL(n,\mathbb{R})$-structure, we denote by S^pM the bundle of symmetric covariant tensor fields of degree p on M. Hence $C^\infty S^pM$, the space of all C^∞-sections of S^pM, is the space symmetric covariant tensor fields of degree p.

Definition 2 (see [18]). A symmetric tensor field $\varphi \in C^\infty S^pM$ on an n-dimensional manifold M with an equiaffine $SL(n,\mathbb{R})$-structure is called Killing tensor of degree p if

$$\varphi\left(\frac{d\gamma}{dt},\dots,\frac{d\gamma}{dt}\right) = \text{const.}$$

along an arbitrary geodesic γ on M.

Let $\varphi\left(\frac{d\gamma}{dt},\dots,\frac{d\gamma}{dt}\right) = \text{const.}$ along an arbitrary geodesic γ on M and hence φ is a Killing tensor. Then the above relation is possible if and only if

$$\delta^*\varphi := \sum_{cicl}\{\nabla\varphi\} = 0$$

where for the local components $\nabla_{i_0}\varphi_{i_1\dots i_p}$ of $\nabla\varphi$ we define the sum

$$\sum_{cicl}\{\nabla_{i_0}\varphi_{i_1\dots i_p}\}$$

as the sum of the terms obtained by a cyclic permutation of indices $i_0, i_1, \dots, i_p$.

Obviously, the set of Killing tensors constitutes an $\mathbb{R}$-module of tensor fields on M, denoted by $\mathbf{T}^p(M,\mathbb{R})$.

Let M be an n-dimensional manifold endowed with an equiprojective $SL(n,\mathbb{R})$-structure (η,∇), and $\omega_1,\dots,\omega_p$ be p linearly independent Killing-Yano tensors of degree 1 on M. Then direct inspection shows that the tensor field $\varphi := \text{sym}\{\omega_1 \otimes \cdots \otimes \omega_p\}$ is a Killing tensor of degree p. Therefore on any n-manifold M with equiprojective $SL(n,\mathbb{R})$-structure, there exist at least $(n+p-1)![p!(n-1)!]^{-1}$ linearly independent Killing tensors (see also [23]). Moreover the following theorem is true.

Theorem 2 *On an n-dimensional manifold M endowed with an equiprojective $SL(n,\mathbb{R})$-structure (η,∇), there exist a local coordinate system $x^1,\dots,x^n$ in which the components $\varphi_{i_1\dots i_p}$ of an arbitrary Killing tensor φ of degree p can be expressed in the form of an p^{th} degree polynomial in the x^i's*

$$\varphi_{i_1\dots i_p} = e^{2p\psi}\sum_{q=0}^{p} A_{i_1\dots i_p j_1\dots j_q} x^{j_1}\dots x^{j_q} \tag{2.2}$$

where the coefficients $A_{i_1\ldots i_p j_1\ldots j_q}$ are constant and symmetric in the set of indices $i_1,\ldots,i_p$ and the set of indices $j_1,\ldots,j_q$. In addition to these properties the coefficients $A_{i_1\ldots i_p j_1\ldots j_q}$ have the following symmetries

$$\sum_{cicl}\{A_{i_1\ldots i_p j_1\ldots j_{p-s}}\}_{j_{p-s+1}} = 0 \tag{2.3}$$

for $s = 1,\ldots,p-1$ and

$$\sum_{cicl}\{A_{i_1\ldots i_p j_1}\} = 0. \tag{2.4}$$

From the theorem we conclude that the maximum number of linearly independent the Killing tensors is obtained by calculating the number T_n^p of independent $A_{i_1\ldots i_p j_1\ldots j_q}$ $(q = 0,1,\ldots,n)$ which exist after accounting for the symmetries on the indices the dependence relations (2.3) and (2.4). By [4] it follows that

$$T_n^p = \frac{p(p+1)^2(p+2)^2\ldots(m+p-1)^2(m+p)}{(p+1)!p!}$$

is the maximum number linearly independent the Killing–Yano tensors. Then we have the following proposition.

Corollary 2 *Let M be an n-dimensional manifold endowed with an equiprojective $SL(n,\mathbb{R})$-structure then*

$$\dim T^p(M,\mathbb{R}) = \frac{p(p+1)^2(p+2)^2\ldots(m+p-1)^2(m+p)}{p!(p+1)!}.$$

3 Proofs of theorems

3.1. We let $f\colon \bar{M} \to M$ denote the mapping of an $\bar{n}$-dimensional manifold $\bar{M}$ endowed with an equiaffine $SL(\bar{n},\mathbb{R})$-structure onto another an n-dimensional manifold M endowed with an equiaffine $SL(n,\mathbb{R})$-structure, and let f_* be the differential of this mapping. For any covariant tensor field ω on M, we can then define the covariant tensor field $f^*\omega$ on $\bar{M}$, where f^* is the transformation transposed to the transformation f_*.

If $\dim\bar{M} = \dim M = n$ and $f\colon \bar{M} \to M$ is a projective diffeomorphism, i.e., a mapping that transforms an arbitrary geodesic in $\bar{M}$ into a geodesic in M, then we have the following lemma.

Lemma 1 *Let $f\colon \bar{M} \to M$ be a projective diffeomorphism of n-dimensional manifolds endowed with the equiaffine $SL(n,\mathbb{R})$-structures $(\bar{\eta},\bar{\nabla})$ and (η,∇) respectively. Then for an arbitrary Killing-Yano tensor ω of degree p $(1 \le p \le n-1)$ on the manifold M the tensor field $\bar{\omega} = e^{-(p+1)\psi}(f^*\omega)$ with $\psi = (n+1)^{-1}\ln(\eta/\bar{\eta})$ will be the Killing-Yano tensor of degree p on the manifold $\bar{M}$.*

Proof It is known that the diffeomorphism $f\colon \bar{M} \to M$ can be realized following the principle of equality of the local coordinates $\bar{x}^1 = x^1, \dots, \bar{x}^n = x^n$ at the corresponding points $\bar{x}$ and $x = f(\bar{x})$ of these manifolds. In this case, we have the equalities (see [15, §18], [9, 10, 26])

$$\Gamma^k_{ij} = \bar{\Gamma}^k_{ij} + \psi_i \delta^k_j + \psi_j \delta^k_i \tag{3.1}$$

for the objects Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$ of the a equiaffine connections ∇ and $\bar{\nabla}$ in the coordinate system $x^1, \dots, x^n$ that is common w.r.t. the mapping $f\colon \bar{M} \to M$, and for the gradient $\psi_j = (n+1)^{-1} \partial_j \ln[\eta/\bar{\eta}]$.

Equalities (3.1) imply that the mapping f^{-1}, which in inverse to the projective diffeomorphism $f\colon \bar{M} \to M$, is a projective mapping [10, p. 262].

We set $\omega_{i_1 \dots i_p}$ be the local components of a Killing-Yano tensor ω of degree p $(1 \le p \le n-1)$ arbitrary defined on the manifold M; by definition, these components satisfy the equations

$$\nabla_{i_0} \omega_{i_1 \dots i_p} + \nabla_{i_1} \omega_{i_0 \dots i_p} = 0. \tag{3.2}$$

From equalities (3.2) we find directly that the components

$$\bar{\omega}_{i_1 \dots i_p} = e^{-(p+1)\psi} \omega_{i_1 \dots i_p} \tag{3.3}$$

of the tensor field $\bar{\omega} = e^{-(p+1)\psi}(f^* \omega)$ satisfy the equations

$$\bar{\nabla}_{i_0} \bar{\omega}_{i_1 \dots i_p} + \bar{\nabla}_{i_1} \bar{\omega}_{i_0 \dots i_p} = 0. \tag{3.4}$$

Hence, the tensor field $\bar{\omega}$ is a Killing-Yano tensor of degree p $(1 \le p \le n-1)$ on the manifold $\bar{M}$. □

3.2. Let $\mathbf{A}^n$ be an n-dimensional affine space with a volume element given by the determinant: $\det(e_1, \dots, e_n) = 1$, where $\{e_1, \dots, e_n\}$ is the standard basis of the underlying vector space for $\mathbf{A}^n$. We denote by ∇ the standard linear connection in $\mathbf{A}^n$ relative to which the volume element "det" is parallel (see [13], [16, p. 10]).

Let $f\colon \bar{M} \to \mathbf{A}^n$ be a projective diffeomorphism from a manifold $\bar{M}$ endowed with equiaffine $SL(n, \mathbb{R})$-structure onto an affine space $\mathbf{A}^n$ endowed with standard equiaffine $\mathrm{S}L(n, \mathbb{R})$-structure. It is well known that manifolds endowed with equiprojective $SL(n, \mathbb{R})$-structures and only these manifolds are projectively diffeomorphic to an affine space $\mathbf{A}^n$ (see [15, §18], [9]) therefore in our case the $SL(n, \mathbb{R})$- structure of the manifold $\bar{M}$ must be an equiprojective structure.

If $\mathbf{A}^n$ is an affine space with the Cartesian system of coordinates $\bar{x}_1, \dots, \bar{x}^n$ then the components $\bar{\omega}_{i_1 \dots i_p}$ of the Killing-Yano tensor $\bar{\omega}$ of degree p $(1 \le p \le n-1)$ in equation (3.4) must now satisfy

$$\partial_j \bar{\omega}_{i i_1 \dots i_p} + \partial_i \bar{\omega}_{j i_1 \dots i_p} = 0 \tag{3.5}$$

where $\partial_j = \frac{\partial}{\partial x^j}$. From (3.5) we conclude the following equations

$$\partial_k \partial_j \bar{\omega}_{i i_1 \dots i_p} + \partial_k \partial_i \bar{\omega}_{j i_1 \dots i_p} = 0; \tag{3.6}$$

$$\partial_j \partial_i \bar{\omega}_{ki_1 \dots i_p} + \partial_j \partial_k \bar{\omega}_{ii_1 \dots i_p} = 0; \qquad (3.7)$$

$$\partial_i \partial_k \bar{\omega}_{ji_1 \dots i_p} + \partial_i \partial_j \bar{\omega}_{ki_1 \dots i_p} = 0. \qquad (3.8)$$

From (3.6), (3.7), (3.8) we find

$$\partial_k \partial_j \bar{\omega}_{i_1 i_2 \dots i_p} = 0, \qquad (3.9)$$

by using identities $\frac{\partial^2 h}{\partial \bar{x}^k \partial \bar{x}^j} = \frac{\partial^2 h}{\partial \bar{x}^j \partial \bar{x}^k}$ which are carried out for an arbitrary smooth function $h\colon \mathbf{A}^n \to \mathbb{R}$. The integrals of equations (3.9) take the form

$$\bar{\omega}_{i_1 \dots i_p} = A_{i_0 i_1 \dots i_p} \bar{x}^{i_0} + B_{i_1 \dots i_p} \qquad (3.10)$$

for any skew-symmetric constants $A_{i_0 i_1 \dots i_p}$ and $B_{i_1 \dots i_p}$ (see also [23, 19]). Taking the components (3.10) of the Killing-Yano tensor $\bar{\omega}$ in $\mathbf{A}^n$ and using Lemma 1, we can formulate Theorem 1.

3.3. Let $\bar{M}$ be a manifold of dimension n endowed with the equiaffine $SL(n, \mathbb{R})$-structure $(\bar{\eta}, \bar{\nabla})$ and M be a manifold of some dimension endowed with the equiaffine $SL(n, \mathbb{R})$-structure (η, ∇). Let there is given a projective diffeomorphism $f\colon \bar{M} \to M$, then we have the following lemma.

Lemma 2 *Let $f\colon \bar{M} \to M$ be a projective diffeomorphism of n-dimensional manifolds endowed with the equiaffine $SL(n, \mathbb{R})$-structures $(\bar{\eta}, \bar{\nabla})$ and (η, ∇) respectively. Then for an arbitrary Killing tensor φ of degree p on the manifold M the tensor field $\bar{\varphi} = e^{-2p\psi}(f^* \varphi)$ with $\psi = (n+1)^{-1} \ln(\eta / \bar{\eta})$ will be the Killing tensor of degree p on the manifold $\bar{M}$.*

Proof We set $\varphi_{i_1 \dots i_p}$ to be components of the Killing tensor φ arbitrary defined on the manifold M; by definition, these components satisfy the following equations $\sum_{cicl} \{\nabla_{i_0} \varphi_{i_1 \dots i_p}\} = 0$. Then we find directly that the components $\bar{\varphi}_{i_1 \dots i_p} = e^{-2p\psi} \varphi_{i_1 \dots i_p}$ of the tensor $\bar{\varphi} = e^{-2p\psi} \varphi$ satisfy the equations

$$\sum_{cicl} \{\bar{\nabla}_{i_0} \bar{\varphi}_{i_1 \dots i_p}\} = e^{-2p\psi} \sum_{cicl} \{\nabla_{i_0} \varphi_{i_1 \dots i_p}\} = 0. \qquad (3.11)$$

From (3.11) we conclude that the tensor field $\bar{\varphi}$ is a Killing tensor of degree p on the manifold $\bar{M}$. □

3.4. It follows from Nijenhuis (see [11]) that in an n-dimensional affine space $\mathbf{A}^n$ the components $\bar{\varphi}_{i_1 \dots i_p}$ of the Killing tensor $\bar{\varphi}$ of degree p can be expressed in the form of an p^{th} degree polynomial in the $\bar{x}^i$'s

$$\varphi_{i_1 \dots i_p} = e^{-2p\psi} \sum_{q=0}^{p} A_{i_1 \dots i_p j_1 \dots j_q} \bar{x}^{j_1} \dots \bar{x}^{j_q}. \qquad (3.12)$$

The coefficients $A_{i_1 \dots i_p j_1 \dots j_q}$ are constant and symmetric in the set of indices $i_1, \dots, i_p$ and the set of indices $j_1, \dots, j_q$. In addition to these properties the coefficients $A_{i_1 \dots i_p j_1 \dots j_q}$ have the following symmetries

$$\sum_{cicl} \{A_{i_1 \dots i_p j_1 \dots j_{p-s}}\}_{j_{p-s+1}} = 0$$

for $s = 1, \dots, p-1$ and

$$\sum_{cicl} \{A_{i_1 \dots i_p j_1}\} = 0.$$

Taking the components (3.12) of the Killing tensor $\bar{\varphi}$ in $\mathbf{A}^n$ and using Lemma 2, we can formulate Theorem 2.

4 Applications to Riemannian geometry

4.1. Let (M, g) be a pseudo-Riemannian manifold of dimensional n. Then from the present theorems 1 and 2 we conclude that an arbitrary Killing vector ω has the following local covariant components $\omega_i = e^{2\psi}(A_{ik}x^k + B_i)$ where $\psi = [2(n+1)]^{-1} \ln |\det g|$, A's and B's are constants and $A_{ik} + A_{ki} = 0$ (see also [17]). It follows that the group of infinitesimal isometric transformations has $\frac{1}{2}n(n+1)$ parameters (see also [2, §71]).

4.2. Following [25, 5], a skew-symmetric covariant tensor field ϑ of degree p $(1 \leq p \leq n-1)$ is called a conformal Killing tensor if $\vartheta \in \ker D$ for

$$D = \nabla - \frac{1}{p+1} d - \frac{1}{n-p+1} g \wedge d^*$$

where d^* is the codifferential operator $d^* \colon C^\infty \Lambda^{p+1} M \to C^\infty \Lambda^p M$ and

$$(g \wedge d^* \vartheta)_{i_0 i_1 \dots i_p} = \sum_{a=1}^{p} (-1)^{a+1} g_{i_0 i_a} (d^* \vartheta)_{i_1 \dots \hat{i}_a \dots i_p}.$$

Obviously, the set of conformal Killing tensors constitutes an vector space of tensor fields on (M, g), denoted by $\mathbf{C}^p(M, \mathbb{R})$ (see [21]). If a conformal Killing tensor ϑ belongs to $ker d^*$, then it is a Killing-Yano tensor. On the other hand, if a conformal Killing tensor ϑ belongs to $ker d$, it is called a closed conformal Killing tensor or a planar tensor (see [20, 21, 22]). We denote the vector space of these tensors by $\mathbf{P}^p(M, \mathbb{R})$.

By [5] on an arbitrary n-dimensional pseudo-Riemannian manifold (M, g) of constant nonzero sectional curvature C $(C \neq 0)$ the vector space $\mathbf{C}^p(M, \mathbb{R})$ of conformal Killing tensors is decomposed uniquely in the form

$$\mathbf{C}^p(M, \mathbb{R}) = \mathbf{K}^p(M, \mathbb{R}) \oplus \mathbf{P}^p(M, \mathbb{R}). \tag{4.1}$$

From (4.1) we conclude that any conformal Killing tensor ϑ of degree p is decomposed uniquely in the form $\vartheta = \omega + \theta$ where ω is a Killing-Yano tensor of degree p and θ is a closed conformal Killing tensor of degree p.

Following theorem 1, on an n-dimensional pseudo-Riemannian manifold (M, g) of constant nonzero sectional curvature C $(C \neq 0)$ there is a local coordinate system $x^1, \dots, x^n$ in which an arbitrary Killing-Yano tensor ω of degree p $(2 \leq p \leq n-1)$ has the components

$$\omega_{i_1 \dots i_p} = e^{(p+1)\psi}(A_{i_0 i_1 \dots i_p} x^{i_0} + B_{i_1 \dots i_p}) \tag{4.2}$$

where $\psi = [2(n+1)]^{-1} \ln|\det g|$, $\psi_k = \frac{\partial\psi}{\partial x^k}$ and $A_{i_0 i_1 \dots i_p}$, $B_{i_1 \dots i_p}$ are arbitrary skew-symmetric constants. On the other hand, by [19] on a pseudo-Riemannian manifold (M, g) of constant nonzero curvature C ($C \neq 0$) the components $\theta_{i_1 \dots i_p}$ of a closed conformal Killing tensor θ of degree p ($1 \leq p \leq n-1$) can be found from the equations

$$\theta_{i_1 i_2 \dots i_p} = -\frac{1}{pC} \nabla_{i_1} \omega_{i_2 \dots i_p} \tag{4.3}$$

where $\nabla_{i_1} \omega_{i_2 \dots i_p} = \partial_{i_1} \omega_{i_2 \dots i_p} - \omega_{k \dots i_p} \Gamma^k_{i_2 i_1} - \dots - \omega_{i_2 \dots k} \Gamma^k_{i_p i_1}$ is the expression for the covariant derivative $\nabla\omega$ of the Killing-Yano tensor of degree $p-1$. Moreover, by virtue of (3.1) on a pseudo-Riemannian manifold (M, g) of constant curvature C ($C \neq 0$) the Christoffel symbols Γ^k_{ij} have the following form $\Gamma^k_{ij} = \psi_i \delta^k_j + \psi_j \delta^k_i$ (see also [17]). Therefore, we can deduce from (4.2) and (4.3) that

$$\theta_{i_1 \dots i_p} = -\frac{1}{C} e^{p\psi} (\psi_{[i_1} A_{|k| i_2 \dots i_p]} x^k + \psi_{[i_1} B_{i_2 \dots i_p]} + \frac{1}{p} A_{i_1 i_2 \dots i_p}).$$

Consequently we have

Theorem 3 *On an n-dimensional pseudo-Riemannian manifold (M, g) of constant nonzero sectional curvature C ($C \neq 0$) there is a local coordinate system $x^1, \dots, x^n$ in which an arbitrary conformal Killing tensor ϑ of degree p ($2 \leq p \leq n-1$) has the components*

$$\vartheta_{i_1 \dots i_p} = e^{(p+1)\psi} (A_{k i_1 \dots i_p} x^k + B_{i-1 \dots i_p})$$
$$-\frac{1}{C} e^{p\psi} \left(\psi_{[i_1} C_{|k| i_2 \dots i_p]} x^k + \psi_{[i_1} D_{i_2 \dots i_p]} + \frac{1}{p} C_{i_1 i_2 \dots i_p} \right)$$

where $\psi = [2(n+1)]^{-1} \ln|\det g|$, $\psi_k = \frac{\partial\psi}{\partial x^k}$ and $A_{i_0 i_1 \dots i_p}$, $B_{i_1 \dots i_p}$, $C_{i_1 \dots i_p}$ and $D_{i_1 \dots i_p}$ are arbitrary skew-symmetric constants.

Remark 3 For a conformal Killing vector field, see K. Yano and T. Nagano [27].

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