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On the Example of Almost Pseudo-Z-symmetric Manifolds*

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Abstract

In the present paper we have obtained a new example of non-Ricci-flat almost pseudo-Z-symmetric manifolds in the class of equidistant spaces, which admit non-trivial geodesic mappings.

Key words: (pseudo-) Riemannian manifold, almost pseudo-Z-symmetric spaces, equidistant spaces.

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1 Introduction

In [4] was introduced an *almost pseudo-Z-symmetric space*, which is an n-dimension (pseudo-) Riemannian space V_n where the special tensor

$$Z_{ij} = R_{ij} + \varphi\, g_{ij},$$

satisfied the recurrent condition

$$Z_{ij,k} = (a_k + b_k)Z_{ij} + a_j Z_{ik} + a_i Z_{jk} \tag{1}$$

R_{ij}, g_{ij} and φ being Ricci tensor, metric tensor and scalar function.

These manifolds are generalization of symmetric and reccurent spaces which were introduced by É. Cartan [2], and A. G. Walker [19], respectively.

These manifolds were generalized in many directions, see, for example [13, pp. 292–295, 335, 338], [18]. Geodesic and holomorphically projective mappings

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of mentioned manifolds were studied in many papers too, see [6, 8, 11, 12, 13, 15, 17]. Among others, J. Mikeš [9] proved that non-Einstein Ricci-symmetric (pseudo-) Riemannian spaces ($R_{ij,k} = 0$) do not admit non-trivial geodesic mappings. In paper [10] were constructed projective symmetric space which is not symmetric. For example, generalized recurrent spaces were studied in [5, 7, 14, 16].

In the paper [4] was studied almost pseudo-Z-symmetric space. As we can see, the Example 8, on p. 39–40, is false for explicit calculation. In this paper, we construct new example of these manifolds.

2 Equidistant manifolds

Having found the example of almost pseudo-Z-symmetric manifolds faulty [4], the present authors have constructed an example in the class of special equidistant space.

In an *equidistant space* with non isotropic concircular vector field there exists canonical coordinate system, where the metric tensor has the following form [17, pp. 92–95], [13, p. 150]:

$$ds^2 = e\, {dx^1}^2 + f(x^1)\; d\tilde{s}^2, \tag{2}$$

where $e = \pm 1$, f is a differentiable function and

$$d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \dots, x^n)\, dx^a dx^b$$

is a metric of $(n-1)$-dimensional (pseudo-) Riemannian manifold $\tilde{V}_{n-1}$.

Here and after indices $a, b, \dots = 2, 3, \dots, n$.

In 1954 N. S. Sinyukov (see [17], [13, pp. 140-155]), thanks to their geometrical properties, gave them the name *equidistant space*. Around the year 1920 the H. W. Brinkmann [1] started studying these space and in the 1940 K. Yano [20] studied concircular vector fields. Many newly obtained results are possible to see in [3].

We denote that if $f' \neq 0$, then this manifold admits non-trivial geodesic mappings, see [17, 11, 13]. In the coordinate system (2) the components of metric and inverse metric tensors have the following form:

$$\begin{aligned} g_{11} &= e; \quad g_{1a} = 0; \quad g_{ab} = f(x^1)\, \tilde{g}_{ab} \\ g^{11} &= e; \quad g^{1a} = 0; \quad g^{ab} = f(x^1)^{-1} \tilde{g}^{ab}, \end{aligned} \tag{3}$$

where f ($\neq 0$) is a function of variable x^1 and $\tilde{g}_{ab}$ and $\tilde{g}^{ab}$ are components of metric and inverse metric tensors of $(n-1)$-dimension on (pseudo-) Riemannian space $\tilde{V}_{n-1}$, their component are functions of variables $x^2, x^3, \dots, x^n$.

Now, non-zero components of Christofell symbols:

$$\Gamma^h_{ij} = \Gamma_{ijk} g^{kh} \quad \text{and} \quad \Gamma_{ijk} = \frac{1}{2} \left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$$

where $\partial_i \equiv \partial/\partial x^i$, have the following form:

$$\Gamma_{1ab} \equiv \Gamma_{a1b} = \frac{1}{2} f' \,\tilde{g}_{ab}; \quad \Gamma_{ab1} = -\frac{1}{2} f' \tilde{g}_{ab}; \quad \Gamma_{abc} = f\,\tilde{\Gamma}_{abc}$$

and non-zero components of Christofell symbols of second kind:

$$\Gamma^1_{ab} = -\frac{e}{2} f' \tilde{g}_{ab}; \quad \Gamma^c_{1b} \equiv \Gamma^c_{b1} = \frac{1}{2}\frac{f'}{f}\delta^c_b; \quad \Gamma^c_{ab} = \tilde{\Gamma}^c_{ab} \tag{4}$$

Following computation of non-zero components of the Riemannian tensor

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k} \tag{5}$$

$$\begin{aligned} R^1_{a1b} &\equiv -R^1_{ab1} = -\tfrac{e}{2}(f'' - \tfrac{f'^2}{2f})\,\tilde{g}_{ab}, \\ R^d_{1b1} &= -\tfrac{1}{2f}(f'' - \tfrac{f'^2}{2f})\delta^c_b\,\tilde{g}_{db}, \\ R^d_{abc} &= \tilde{R}^d_{abc} - \tfrac{e}{4}\tfrac{f'^2}{f}(\tilde{g}_{ac}\,\delta^d_b - \tilde{g}_{ab}\;\delta^d_c). \end{aligned}$$

Contracting Riemannian tensor by metric tensor, we lower indices and obtain Riemannian tensor of type $\binom{0}{4}$

$$R_{hijk} = g_{h\alpha} R^\alpha_{ijk}. \tag{6}$$

After computation, we get the following non-zero components:

$$\begin{aligned} R_{1a1b} &= -R_{a11b} = R_{a1b1} = R_{a11b} = -\tfrac{1}{2}(f'' - \tfrac{f'^2}{2f})\tilde{g}_{ab} \\ R_{abcd} &= f\tilde{R}_{abcd} - \tfrac{e}{4} f'^2(\tilde{g}_{ac}\tilde{g}_{bd} - \tilde{g}_{ad}\tilde{g}_{ac}). \end{aligned}$$

The Ricci tensor $R_{ij} = R^\alpha_{i\alpha j}$ has these non-zero components:

$$\begin{aligned} R_{11} &= R^\alpha_{1\alpha 1} = -\tfrac{1}{2f}(n-1)(f'' - \tfrac{f'^2}{2f}) \\ R_{ab} &= \tilde{R}_{ab} - \tfrac{e}{2}\,(f'' - \tfrac{f'^2}{2f})\tilde{g}_{ab}. \end{aligned}$$

3 Special equidistant almost pseudo-Z-symmetric spaces

The above mentioned almost pseudo-Z-symmetric spaces are defined in formula (1). Next, we shall study these spaces supposing that this space V_n is equidistant, and moreover $\tilde{V}_{n-1}$ is Ricci flat space and component Z_{11} of tensor Z is equal to zero.

Firstly, we compute non-zero components of tensor $Z_{ij} = R_{ij} + \varphi g_{ij}$:

$$\begin{aligned} Z_{11} &= R_{11} + \varphi(x^1)g_{11} = -\tfrac{1}{2f}(n-1)(f'' - \tfrac{f'^2}{2f}) + e\varphi; \\ Z_{ab} &= -(\tfrac{e}{2}(f'' - \tfrac{f'^2}{2f}) - \varphi f)\tilde{g}_{ab}. \end{aligned}$$

From our proposition ($Z_{11} = 0$) it follows that the function φ has the following form:

$$\varphi = \frac{e}{2}(n-1)\left(f'' - \frac{f'^2}{2f}\right), \tag{7}$$

and thus

$$Z = -\frac{en}{2}\left(f'' - \frac{f'^2}{2f}\right). \tag{8}$$

Secondly, we remember that covariant derivations of Z_{ij} have the following definition

$$Z_{ij,k} = \partial_k Z_{ij} - Z_{\alpha j}\Gamma^{\alpha}_{ik} - Z_{i\alpha}\Gamma^{\alpha}_{jk},$$

and equation (1):

$$Z_{ij,k} = (a_k + b_k)Z_{ij} + a_j Z_{ik} + a_i Z_{jk}$$

will have the form

$$\begin{aligned}
&Z_{11,1} \equiv \partial_1 Z_{11} = (3a_1 + b_1)Z_{11};\\
&Z_{11,c} \equiv 0 = (a_c + b_c)Z_{11};\\
&Z_{1b,1} \equiv 0 = a_b Z_{11};\\
&Z_{1b,c} \equiv -\tfrac{f'}{2f}\, Z_{bc} + \tfrac{e}{2}\, f' Z_{11}\tilde{g}_{bc} = a_1 Z_{bc};\\
&Z_{ab,1} \equiv \partial_1 Z_{ab} - \tfrac{f'}{f}\, Z_{ab} = (a_1 + b_1)Z_{ab};\\
&Z_{ab,c} \equiv 0 = (a_c + b_c)Z_{ab} + a_a Z_{bc} + a_b Z_{ac}.
\end{aligned}$$

Because $Z_{11} = 0$, the above equations are simplify to the following form:

$$-\frac{f'}{2f}\, Z_{bc} = a_1 Z_{bc}; \tag{9}$$

$$\partial_1 Z_{ab} - \frac{f'}{f}\, Z_{ab} = (a_1 + b_1)Z_{ab}; \tag{10}$$

$$(a_c + b_c)Z_{ab} + a_a Z_{bc} + a_b Z_{ac} = 0. \tag{11}$$

Naturally $Z_{ij} \neq 0$, then Z must not be equal to zero. Then for $n \geq 4$ and from (11) it implies $a_a = b_a = 0$. From (9) and (10) follows:

$$a_1 = -\frac{1}{2}\frac{f'}{f}, \quad \text{and} \quad b_1 = -a_1 - \frac{f'}{f}\partial_1 \ln |Z|.$$

On the base of above discussion, we can formulate this theorem:

Theorem 1 *The equidistant space with metric* (2) *where metric $d\tilde{s}^2$ defined Ricci-flat space is almost pseudo-Z-symmetric space for any non-zero function $f(x^1) \in C^3$, $f'' - \frac{f'^2}{2f} \neq 0$.*

In this space we have tensor $Z_{ij} = R_{ij} - \varphi\, g_{ij}$, where

$$\varphi = \frac{e(n-1)}{2}\left(f'' - \frac{f'^2}{2f}\right),$$

and

$$a_i = -\delta_i^1 \left(\frac{f'}{2f}\right) \quad \text{and} \quad b_i = -\delta_i^1 \frac{f'}{2f}\left(1 - 2\left(\ln\left|f'' - \frac{f'^2}{2f}\right|\right)'\right).$$

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