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Some Properties of Lorentzian α-Sasakian Manifolds with Respect to Quarter-symmetric Metric Connection

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Abstract

The aim of this paper is to study generalized recurrent, generalized Ricci-recurrent, weakly symmetric and weakly Ricci-symmetric, semi-generalized recurrent, semi-generalized Ricci-recurrent Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection. Finally, we give an example of 3-dimensional Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection.

Key words: Quarter-symmetric metric connection, Lorentzian α-Sasakian manifold, generalized recurrent manifold, generalized Ricci-recurrent manifold, weakly symmetric manifold, weakly Ricci-symmetric manifold, semi-generalized recurrent manifold, Einstein manifold.

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1 Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [5]. Further, Hayden [7], introduced the idea of metric connection with torsion on a Riemannian manifold. In [32], Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds.

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In 1975, Golab [6] defined and studied a quarter-symmetric connection in a differentiable manifold.
A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a *quarter-symmetric connection* [6] if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \tag{1.1}$$

is of the form

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a non-zero 1-form and ϕ is a tensor field of type $(1,1)$. In addition, if a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{1.3}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$ for all $X, Y \in \chi(M)$, then the quarter-symmetric connection reduces to a semi-symmetric connection [5].

M. M. Tripathi [29] studied semi-symmetric metric connections in a Kenmotsu manifolds. In [31], the semi-symmetric non-metric connection in a Kenmotsu manifold was studied by M. M. Tripathi and N. Nakkar. Also in [30], M. M. Tripathi proved the existence of a new connection and showed that in particular cases, this connection reduces to semi-symmetric connections; even some of them are not introduced so far.

In 2005, Yildiz and Murathan [36] studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere. In 2012, Yadav and Suthar [34] studied Lorentzian α-Sasakian manifolds.

After Golab [6], Rastogi ([22], [23]) continued the systematic study of quarter-symmetric metric connection. In 1980, Mishra and Pandey [8] studied quarter-symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifold. In 1982, Yano and Imai [33] studied quarter-symmetric metric connection in Hermition and Kaehlerian manifolds. In 1991, Mukhopadhyay et al. [16] studied quarter-symmetric metric connection on a Riemannian manifold with an almost complex structure ϕ.

On the other hand, De and Guha introduced generalized recurrent manifold with the non-zero 1-form α_1 and another non-zero associated 1-form β_1. Such a manifold has been denoted by GK_n. If the associated 1-form becomes zero, then the manifold GK_n reduces to a recurrent manifold introduced by Ruse [24] which is denoted by K_n. The idea of Ricci-recurrent manifold was introduced by Patterson [17]. He denoted such a manifold by R^n. Ricci-recurrent manifolds have been studied by many authors ([3], [18], [35], [9], [10], [11], [12]).

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *generalized recurrent* if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y, Z)W = \alpha_1(X)R(Y, Z)W + \beta_1(X)[g(Z, W)Y - g(Y, W)Z], \quad (1.4)$$

where ∇ is the Levi-Civita connection and α_1 and β_1 are two 1-forms ($\beta_1 \neq 0$) defined by

$$\alpha_1(X) = g(X, A), \;\; \beta_1(X) = g(X, B), \quad (1.5)$$

and A, B are vector fields related with 1-forms α_1 and β_1 respectively. A non-flat n-dimensional differentiable manifold M, $n > 3$, is called generalized Ricci-recurrent if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y, Z)W = \alpha_1(X)S(Y, Z)W + (n - 1)\beta_1(X)g(Y, Z), \quad (1.6)$$

where α_1 and β_1 defined as (1.5).

The notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamassy and T. Q. Binh in ([27], [28]).

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called *pseudosymmetric* if there is a 1-form α_1 on M such that

$$\begin{aligned}(\nabla_X R)(Y, Z)V \;&=\; 2\alpha_1(X)R(Y, Z)V + \alpha_1(Y)R(X, Z)V + \alpha_1(Z)R(Y, X)V \\ &+\; \alpha_1(V)R(Y, Z)X + g(R(Y, Z)V, X)A, \end{aligned} \quad (1.7)$$

where ∇ is the Levi-Civita connection and X, Y, Z, V are vector fields on M. $A \in \chi(M)$ is the vector field associated with 1-form α_1 which is defined by $g(X, A) = \alpha_1(X)$ in [1]. Later R. Deszcz [4] started to use "pseudosymmetric" term in different sence, see([11], [12] [13]).

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called weakly symmetric ([27], [28]) if there are 1-forms α_1, β_1, γ_1, σ_1 such that

$$\begin{aligned}(\nabla_X R)(Y, Z)V \;&=\; \alpha_1(X)R(Y, Z)V + \beta_1(Y)R(X, Z)V + \gamma_1(Z)R(Y, X)V \\ &+\; \sigma_1(V)R(Y, Z)X + g(R(Y, Z)V, X)A \end{aligned} \quad (1.8)$$

for all vector fields X, Y, Z, V on M. A weakly symmetric manifold M is pseudosymmetric if $\beta_1 = \gamma_1 = \sigma_1 = \frac{1}{2}\alpha_1$ and $P = A$, locally symmetric if $\alpha_1 = \beta_1 = \gamma_1 = \sigma_1 = 0$ and $P = 0$. A weakly symmetric manifold is said to be proper if at least one of the 1-forms α_1, β_1, γ_1 and σ_1 is not zero or $P \neq 0$.

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called weakly Ricci-symmetric ([27], [28]) if there are 1-forms ρ, μ, υ such that

$$(\nabla_X S)(Y, Z) = \rho(X)S(Y, Z) + \mu(Y)S(Y, Z) + \upsilon(Z)S(X, Y) \quad (1.9)$$

for all vector fields X, Y, Z, V on M. If $\rho = \mu = \upsilon$, then M is called pseudo Ricci-symmetric(see [2]).
If M is weakly symmetric, from (1.8), we have

$$(\nabla_X S)(Y, Z) = \alpha_1(X)S(Z, V) + \beta_1(R(X, Z)V) + \gamma_1(Z)S(X, V) + \sigma_1(V)S(X, Z) + p(R(X, V)Z), \tag{1.10}$$

where p is defined by $p(X) = g(X, P)$ for any $X \in \chi(M)$ in [28].

Generalizing the notion of recurrency, the author Khan [21] introduced the notion of generalized recurrent Sasakian manifolds. In the paper B. Prasad [19] introduced the notion of semi-generalized recurrent manifold and obtained few interesting results. L. Rachůnek and J. Mikeš studied the similar problems in([14], [15], [25]).

A Riemannian manifold is called a *semi-generalized recurrent manifold* if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y, Z)W = \alpha_1(X)R(Y, Z)W + \beta_1(X)g(Z, W)Y, \tag{1.11}$$

where α_1 and β_1 defined as (1.5).

A Riemannian manifold is called a semi-generalized Ricci-recurrent manifold if its curvature tensor R satisfies the condition

$$(\nabla_X S)(Y, Z) = \alpha_1(X)S(Y, Z) + n\beta_1(X)g(Y, Z), \tag{1.12}$$

where α_1 and β_1 defined as (1.5).

Motivated by the above studies, in the present paper we have proved that $\beta_1 = (\alpha - \alpha^2)\alpha_1$ holds on both generalized recurrent and generalized Ricci-recurrent Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection. We also show that there is no weakly symmetric or weakly Ricci-symmetric Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection, $n > 3$, unless $\alpha_1+\sigma_1+\gamma_1$ or $\rho+\mu+\upsilon$ is everywhere zero, respectively. We have also studied semi-generalized recurrent Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection.

2 Preliminaries

A n(=2m+1)-dimensional differentiable manifold M is said to be a Lorentzian α-Sasakian manifold if it admits a $(1, 1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy the following conditions

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \phi\xi = 0, \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi + \eta(Y)X\} \tag{2.5}$$

$\forall X, Y \in \chi(M)$ and for non-zero smooth functions α on M, ∇ denotes the covariant differentiation with respect to Lorentzian metric g ([20], [37]).
For a Lorentzian α-Sasakian manifold, it can be shown that ([20], [37]):

$$\nabla_X \xi = \alpha\phi X, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = \alpha g(\phi X, Y) \tag{2.7}$$

for all $X, Y \in \chi(M)$.
Further on a Lorentzian α-Sasakian manifold, the following relations hold [20]

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{2.8}$$

$$R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X], \tag{2.9}$$

$$R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi, X)\xi = \alpha^2[X + \eta(X)\xi], \tag{2.11}$$

$$S(X, \xi) = S(\xi, X) = (n-1)\alpha^2\eta(X), \tag{2.12}$$

$$S(\xi, \xi) = -(n-1)\alpha^2, \tag{2.13}$$

$$Q\xi = (n-1)\alpha^2\xi, \tag{2.14}$$

where Q is the Ricci operator, i.e.,

$$g(QX, Y) = S(X, Y). \tag{2.15}$$

If ∇ is the Levi-Civita connection manifold M, then quarter-symmetric metric connection $\tilde{\nabla}$ in M is denoted by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi(X). \tag{2.16}$$

3 Curvature tensor and Ricci tensor of Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

Let $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ be the curvature tensors with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and with respect to the Riemannian connection ∇ respectively on a Lorentzian α-Sasakian manifold M. A relation between the curvature tensors $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ on M is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] \\ &+ \alpha\eta(Z)[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.1}$$

Also from (3.1), we obtain

$$\tilde{S}(X,Y) = S(X,Y) + \alpha[g(X,Y) + n\eta(X)\eta(Y)], \tag{3.2}$$

where $\tilde{S}$ and S are the Ricci tensor with respect to $\tilde{\nabla}$ and ∇ respectively.
Contracting (3.2), we obtain,

$$\tilde{r} = r, \tag{3.3}$$

where $\tilde{r}$ and r are the scalar curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively.
Also we have

$$\tilde{R}(\xi,X)Y = -\tilde{R}(X,\xi)Y = \alpha^2[g(X,Y))\xi - \eta(Y)X] + \alpha\eta(Y)[X + \eta(X)\xi], \tag{3.4}$$

$$\eta(\tilde{R}(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{3.5}$$

$$\tilde{R}(X,Y)\xi = (\alpha^2 - \alpha)[\eta(Y)X - \eta(X)Y], \tag{3.6}$$

$$\tilde{S}(X,\xi) = \tilde{S}(\xi,X) = (n-1)(\alpha^2 - \alpha)\eta(X), \tag{3.7}$$

$$\tilde{S}(\xi,\xi) = -(n-1)(\alpha^2 - \alpha), \tag{3.8}$$

$$\tilde{Q}X = QX - \alpha(n-1)X, \tag{3.9}$$

$$\tilde{Q}\xi = (n-1)(\alpha^2 - \alpha)\xi, \tag{3.10}$$

$$\tilde{R}(\xi,X)\xi = (\alpha^2 - \alpha)[X + \eta(X)\xi]. \tag{3.11}$$

4 Generalized recurrent Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called generalized recurrent with respect to the quarter-symmetric metric connection if its curvature tensor $\tilde{R}$ satisfies the condition

$$(\tilde{\nabla}_X \tilde{R})(Y,Z)W = \alpha_1(X)\tilde{R}(Y,Z)W + \beta_1(X)[g(Z,W)Y - g(Y,W)Z] \quad (4.1)$$

for all $X, Y, Z, W \in \chi(M)$, where $\tilde{\nabla}$ is the quarter-symmetric metric connection and $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$.

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called generalized Ricci-recurrent with respect to the quarter-symmetric metric connection if its Ricci tensor $\tilde{S}$ satisfies the condition

$$(\tilde{\nabla}_X \tilde{S})(Y,Z) = \alpha_1(X)\tilde{S}(Y,Z) + (n-1)\beta_1(X)g(Y,Z) \quad (4.2)$$

for all $X, Y, Z \in \chi(M)$.
In [26] Sular studied that if M be a generalized recurrent Kenmotsu manifold and generalized Ricci recurrent Kenmotsu manifold respect to semi-symmetric metric connection, then $\beta_1 = 2\alpha_1$ holds on M.

Now we consider generalized recurrent and generalized Ricci-recurrent Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection.
We start with the following theorem:

Theorem 4.1. *If a generalized recurrent Lorentzian α-Sasakian manifold M admits quarter-symmetric metric connection, then $\beta_1 = (\alpha - \alpha^2)\alpha_1$ holds on M.*

Proof. Suppose that M is a generalized recurrent Lorentzian α-Sasakian manifold admitting a quarter-symmetric metric connection. Taking $Y = W = \xi$ in (4.1), we get

$$(\tilde{\nabla}_X \tilde{R})(\xi,Z)\xi = \alpha_1(X)\tilde{R}(\xi,Z)\xi + \beta_1(X)[g(Z,\xi)\xi + Z]. \quad (4.3)$$

By using the equation (2.4), (2.10) and (3.6) in (4.3), we have

$$(\tilde{\nabla}_X \tilde{R})(\xi,Z)\xi = [\alpha_1(X)(\alpha^2 - \alpha) + \beta_1(X)]\{\eta(Z)\xi + Z\}. \quad (4.4)$$

On the other hand, it is clear that

$$(\tilde{\nabla}_X \tilde{R})(\xi,Z)\xi = \tilde{\nabla}_X \tilde{R}(\xi,Z)\xi - \tilde{R}(\tilde{\nabla}_X \xi, Z) - \tilde{R}(\xi, \tilde{\nabla}_X Z) - \tilde{R}(\xi,Z)\tilde{\nabla}_X \xi. \quad (4.5)$$

Now using the equation (2.10) and (3.6) in (4.5), we have

$$(\tilde{\nabla}_X \tilde{R})(\xi,Z)\xi = 0. \quad (4.6)$$

Hence comparing the right hand sides of the equations (4.4) and (4.6) we obtain

$$[\alpha_1(X)(\alpha^2-\alpha)+\beta_1(X)]\{\eta(Z)\xi+Z\}=0, \tag{4.7}$$

which imply

$$\beta_1(X)=(\alpha-\alpha^2)\alpha_1(X) \tag{4.8}$$

for any vector field $X \in M$. So our theorem is proved.

Theorem 4.2. *Let M be a generalized Ricci-recurrent Lorentzian α-Sasakian manifold admitting quarter-symmetric metric connection, then $\beta_1=(\alpha-\alpha^2)\alpha_1$ holds on M.*

Proof. Suppose that M is a generalized Ricci-recurrent Lorentzian α-Sasakian Manifold M with respect to quarter-symmetric metric connection. Now putting $Z=\xi$ in (4.2), we get

$$(\tilde{\nabla}_X\tilde{S})(Y,\xi)=\alpha_1(X)\tilde{S}(Y,\xi)+(n-1)\beta_1(X)g(Y,\xi). \tag{4.9}$$

Then by using the equation (2.4), (2.12) and (3.7) in (4.9), we have

$$(\tilde{\nabla}_X\tilde{S})(Y,\xi)=\alpha_1(X)[(n-1)(\alpha^2-\alpha)\eta(Y)+(n-1)\beta_1(X)\eta(Y). \tag{4.10}$$

On the other hand, by using the definition of covariant derivative of $\tilde{S}$ with respect to the quarter-symmetric metric connection, it is well-known that

$$(\tilde{\nabla}_X\tilde{S})(Y,\xi)=\tilde{\nabla}_X\tilde{S}(Y,\xi)-\tilde{S}(\tilde{\nabla}_XY,\xi)-\tilde{S}(Y,\tilde{\nabla}_X\xi) \tag{4.11}$$

Now using the equation (2.6), (2.7), (2.12), (2.16), (3.2) and (3.7) in (4.11), we obtain

$$(n-1)(\alpha^2-\alpha)\alpha g(Y,\phi X)-(\alpha-1)[S(Y,\phi X)+\alpha g(Y,\phi X)]. \tag{4.12}$$

Hence comparing the right hand sides of the equations (4.10) and (4.12) we obtain

$$\begin{aligned}\alpha_1(X)[(n-1)(\alpha^2 &- \alpha)\eta(Y)+(n-1)\beta_1(X)\eta(Y)\\ &= (n-1)(\alpha^2-\alpha)\alpha g(Y,\phi X)\\ &- (\alpha-1)[S(Y,\phi X)+\alpha g(Y,\phi X)].\end{aligned} \tag{4.13}$$

Now putting $Y=\xi$ in (4.13), we get

$$\beta_1(X)=(\alpha-\alpha^2)\alpha_1(X) \tag{4.14}$$

for any vector field $X \in M$. So this completes the proof.

5 Weakly symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called weakly symmetric with respect to quarter-symmetric metric connection if there are 1-forms α_1, β_1, γ_1, σ_1 such that

$$(\tilde{\nabla}_X \tilde{R})(Y,Z)V = \alpha_1(X)\tilde{R}(Y,Z)V + \beta_1(Y)\tilde{R}(X,Z)V + \gamma_1(Z)\tilde{R}(Y,X)V + \sigma_1(V)\tilde{R}(Y,Z)X + g(\tilde{R}(Y,Z)V,X)A \quad (5.1)$$

for all vector fields X, Y, Z, V on M.

A non-flat n-dimensional differentiable manifold M, $n > 3$, is called weakly Ricci-symmetric with respect to quarter-symmetric metric connection if there are 1-forms ρ, μ, υ such that

$$(\tilde{\nabla}_X \tilde{S})(Y,Z) = \rho(X)\tilde{S}(Y,Z) + \mu(Y)\tilde{S}(Y,Z) + \upsilon(Z)\tilde{S}(X,Y) \quad (5.2)$$

for all vector fields X, Y, Z, V on M. If M is weakly symmetric with respect to the quarter-symmetric metric connection, by a contraction from (1.8), we have

$$(\tilde{\nabla}_X \tilde{S})(Z,V) = \alpha_1(X)\tilde{S}(Z,V) + \beta_1(\tilde{R}(X,Z)V) + \gamma_1(Z)\tilde{S}(X,V) + \sigma_1(V)\tilde{S}(X,Z) + p(\tilde{R}(X,V)Z). \quad (5.3)$$

In [26], Sular studied weakly symmetric and weakly Ricci-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection and obtained some results.
i) If M be a weakly symmetric Kenmotsu manifold with respect to quarter-symmetric metric connection then there is no weakly symmetric $n > 3$, unless $\alpha_1 + \sigma_1 + \gamma_1$ is everywhere zero.
ii) If M be a weakly Ricci-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection then there is no weakly Ricci-symmetric $n > 3$, unless $\rho + \mu + \upsilon$ is everywhere zero.

Now we consider weakly symmetric and weakly Ricci-symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection.
We start with the following theorem:

Theorem 5.1. *There is no weakly symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection $n > 3$, unless $\alpha_1 + \sigma_1 + \gamma_1$ is everywhere zero, provided $\alpha \neq 0, 1$.*

Proof. Let M be a weakly symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$. By the covariant differentiation of the Ricci tensor $\tilde{S}$ of the quarter-symmetric metric connection with respect to X, we have

$$(\tilde{\nabla}_X \tilde{S})(Z,V) = \tilde{\nabla}_X \tilde{S}(Z,V) - \tilde{S}(\tilde{\nabla}_X Z, V) - \tilde{S}(Z, \tilde{\nabla}_X V). \quad (5.4)$$

Putting $V = \xi$ in (5.4) and using (2.6), (2.7), (2.12), (2.16) and (3.7), it follows that

$$(\tilde{\nabla}_X \tilde{S})(Z, \xi) = (n-1)(\alpha^2 - \alpha)(\nabla_X \eta)Z - (\alpha - 1)\tilde{S}(Z, \phi X). \tag{5.5}$$

Replacing $V = \xi$ in (5.3), we get

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z, \xi) &= \alpha_1(X)\tilde{S}(Z, \xi) + \beta_1(\tilde{R}(X, Z)\xi) + \gamma_1(Z)\tilde{S}(X, \xi)\\ &+ \sigma_1(\xi)\tilde{S}(X, Z) + p(\tilde{R}(X, \xi)Z).\end{aligned} \tag{5.6}$$

Now using (2.6), (2.7), (2.12), (2.16), (3.6) and (3.7) in (5.6), we obtain

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z, \xi) &= \alpha_1(X)(n-1)(\alpha^2 - \alpha)\eta(Z)\\ &+ (\alpha^2 - \alpha)[\eta(Z)\beta_1(X) - \eta(X)\beta_1(Z)]\\ &+ \gamma_1(Z)(n-1)(\alpha^2 - \alpha)\eta(X) + \sigma_1(\xi)\tilde{S}(X, Z)\\ &- \alpha^2[g(X, Z)p(\xi) - \eta(Z)p(X)] - \alpha_1\eta(Z)[p(X)\\ &+ \eta(X)p(\xi)].\end{aligned} \tag{5.7}$$

Thus, comparing the right hand sides of the equations (5.5) and (5.7) we obtain

$$\begin{aligned}(n-1)(\alpha^2 - \alpha)(\nabla_X \eta)Z &- (\alpha - 1)\tilde{S}(Z, \phi X) = \alpha_1(X)(n-1)(\alpha^2 - \alpha)\eta(Z)\\ &+ (\alpha^2 - \alpha)[\eta(Z)\beta_1(X) - \eta(X)\beta_1(Z)]\\ &+ \gamma_1(Z)(n-1)(\alpha^2 - \alpha)\eta(X) + \sigma_1(\xi)\tilde{S}(X, Z)\\ &- \alpha^2[g(X, Z)p(\xi) - \eta(Z)p(X)] - \alpha_1\eta(Z)[p(X)\\ &+ \eta(X)p(\xi)].\end{aligned} \tag{5.8}$$

Then taking $X = Z = \xi$ in (5.8) and using (2.1), (2.2), (2.4), (2.12) and (3.8), we get

$$(n-1)(\alpha^2 - \alpha)[\alpha_1(\xi) + \gamma_1(\xi) + \sigma_1(\xi)] = 0. \tag{5.9}$$

Now as $n > 3$ and $\alpha \neq 0, 1$, So,

$$\alpha_1(\xi) + \gamma_1(\xi) + \sigma_1(\xi) = 0. \tag{5.10}$$

Now putting $Z = \xi$ in (5.3), we get

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(\xi, V) &= \alpha_1(X)\tilde{S}(\xi, V) + \beta_1(\tilde{R}(X, \xi)V) + \gamma_1(\xi)\tilde{S}(X, V)\\ &+ \sigma_1(\xi)\tilde{S}(X, \xi) + p(\tilde{R}(X, V)\xi.\end{aligned} \tag{5.11}$$

Also putting $Z = \xi$ in (5.4) and using (2.6), (2.7), (2.12), (2.16) and (3.7), it follows that

$$(\tilde{\nabla}_X \tilde{S})(\xi, V) = (n-1)(\alpha^2 - \alpha)(\nabla_X \eta)V - (\alpha - 1)\tilde{S}(V, \phi X). \tag{5.12}$$

Similarly using (2.6), (2.7), (2.12), (2.16), (3.6) and (3.7) in (5.11), we obtain

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(\xi, V) &= \alpha_1(X)(n-1)(\alpha^2 - \alpha)\eta(V) - \alpha^2[g(X, V)\beta_1(\xi)\\ &- \eta(V)\beta_1(X)] - \alpha\eta(V)[\beta_1(X) + \eta(X)\beta_1(\xi)]\\ &+ \gamma_1(\xi)\tilde{S}(X, V) + \sigma_1(V)(n-1)(\alpha^2 - \alpha)\eta(X)\\ &+ (\alpha^2 - \alpha)[\eta(V)p(X) - \eta(V)p(X)].\end{aligned} \tag{5.13}$$

Thus, comparing the right hand sides of the equations (5.12) and (5.13), we obtain

$$\begin{aligned}(n-1)(\alpha^2-\alpha)(\nabla_X\eta)V \;&-\; (\alpha-1)\tilde{S}(V,\phi X) = \alpha_1(X)(n-1)(\alpha^2-\alpha)\eta(V)\\ &-\; \alpha^2[g(X,V)\beta_1(\xi)-\eta(V)\beta_1(X)] - \alpha\eta(V)[\beta_1(X)\\ &+\; \eta(X)\beta_1(\xi)] + \gamma_1(\xi)\tilde{S}(X,V) + \sigma_1(V)(n-1)(\alpha^2\\ &-\; \alpha)\eta(X) + (\alpha^2-\alpha)[\eta(V)p(X)\\ &-\; \eta(V)p(X)]. \qquad (5.14)\end{aligned}$$

Now putting $V=\xi$ in (5.14), we obtain

$$\begin{aligned}-\alpha_1(X)(n-1)(\alpha^2-\alpha) \;&-\; (\alpha^2-\alpha)[\eta(X)\beta_1(\xi)+\beta_1(X)]\\ &+\; (\sigma_1(\xi)+\gamma_1(\xi))(n-1)(\alpha^2-\alpha)\eta(X)\\ &-\; (\alpha^2-\alpha)[p(X)+\eta(X)p(\xi)] = 0. \qquad (5.15)\end{aligned}$$

Taking $X=\xi$ in (5.14), we obtain

$$\begin{aligned}\alpha_1(\xi)(n-1)(\alpha^2-\alpha)\eta(V) \;&+\; \gamma_1(\xi)(n-1)(\alpha^2-\alpha)\eta(V)\\ &-\; \sigma_1(V)(n-1)(\alpha^2-\alpha) + (\alpha^2\\ &-\; \alpha)[p(V)+\eta(V)p(\xi)] = 0. \qquad (5.16)\end{aligned}$$

In (5.16) taking $V=X$ and summing with (5.15), by virtue of (5.10) we find

$$\begin{aligned}-(n-1)(\alpha^2-\alpha)[\alpha_1(X) \;&+\; \sigma_1(X)] - (\alpha^2-\alpha)[\eta(X)\beta_1(\xi)+\beta_1(X)]\\ &+\; (n-1)(\alpha^2-\alpha)\eta(X)\gamma_1(\xi) = 0. \qquad (5.17)\end{aligned}$$

Again putting $X=\xi$ in (5.8), we obtain

$$\begin{aligned}\alpha_1(\xi)(n-1)(\alpha^2-\alpha)\eta(Z) \;&+\; (\alpha^2-\alpha)[\eta(Z)\beta_1(\xi)+\beta_1(Z)]\\ &-\; \gamma_1(Z)(n-1)(\alpha^2-\alpha)\\ &+\; \sigma_1(\xi)(n-1)(\alpha^2-\alpha)\eta(Z) = 0. \qquad (5.18)\end{aligned}$$

Now in the equation (5.18) taking $Z=X$, we obtain

$$\begin{aligned}\alpha_1(\xi)(n-1)(\alpha^2-\alpha)\eta(X) \;&+\; (\alpha^2-\alpha)[\eta(X)\beta_1(\xi)+\beta_1(X)]\\ &-\; \gamma_1(X)(n-1)(\alpha^2-\alpha)\\ &+\; \sigma_1(\xi)(n-1)(\alpha^2-\alpha)\eta(X) = 0. \qquad (5.19)\end{aligned}$$

Then adding (5.17) and (5.19) , we find

$$\begin{aligned}(n-1)(\alpha^2-\alpha)\eta(X)[\alpha_1(\xi) \;&+\; \gamma_1(\xi)+\sigma_1(\xi)] - (n-1)(\alpha^2-\alpha)[\alpha_1(X)\\ &+\; \gamma_1(X)+\sigma_1(X)] = 0. \qquad (5.20)\end{aligned}$$

Since $n>3$, $\alpha\neq 0,1$, and

$$\alpha_1(\xi)+\gamma_1(\xi)+\sigma_1(\xi) = 0,$$

so we get

$$\alpha_1(X) + \gamma_1(X) + \sigma_1(X) = 0$$

for all $X \in M$.
So our proof is completed.

Theorem 5.2. *There is no weakly Ricci-symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection $n > 3$, unless $\rho + \mu + \upsilon$ is everywhere zero, provided $\alpha \neq 0, 1$.*

Proof. Assume that M is a weakly Ricci-symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$. Now taking $Z = \xi$ in (5.2) and using (3.2) and (3.7), we obtain

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Y, \xi) &= \rho(X)(n-1)(\alpha^2 - \alpha)\eta(Y) + \mu(X)(n-1)(\alpha^2 - \alpha)\eta(X) \\ &+ \upsilon(\xi)[S(X, Y) + \alpha\{g(X, Y) + n\eta(X)\eta(Y)\}]. \end{aligned} \tag{5.21}$$

Also we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Y, \xi) &= (n-1)(\alpha^2 - \alpha)(\nabla_X \eta)(Y) - (\alpha - 1)[S(Y, \phi X) \\ &+ \alpha g(X, \phi Y)]. \end{aligned} \tag{5.22}$$

Now equating (5.21) and (5.22), we obtain

$$\begin{aligned}\rho(X)(n-1)(\alpha^2 - \alpha)\eta(Y) &+ \mu(X)(n-1)(\alpha^2 - \alpha)\eta(X) + \upsilon(\xi)[S(X, Y) \\ &+ \alpha\{g(X, Y) + n\eta(X)\eta(Y)\}] = (n-1)(\alpha^2 \\ &- \alpha)(\nabla_X \eta)(Y) - (\alpha - 1)[S(Y, \phi X) \\ &+ \alpha g(X, \phi Y)]. \end{aligned} \tag{5.23}$$

Now putting $X = Y = \xi$ in (5.23), we find

$$(n-1)(\alpha^2 - \alpha)[\rho(\xi) + \mu(\xi) + \upsilon(\xi)] = 0. \tag{5.24}$$

As $n > 3$ and $\alpha \neq 0, 1$, So

$$\rho(\xi) + \mu(\xi) + \upsilon(\xi) = 0. \tag{5.25}$$

Taking $X = \xi$ in (5.23), we find

$$(n-1)(\alpha^2 - \alpha)\eta(Y)[\rho(\xi) + \upsilon(\xi)] + \mu(Y)(n-1)(\alpha^2 - \alpha) = 0. \tag{5.26}$$

So in view of (5.25), the above equation turns into

$$-\eta(Y)\mu(\xi) = \mu(Y). \tag{5.27}$$

Similarly in (5.23), taking $Y = \xi$, we find

$$-\rho(X)(n-1)(\alpha^2 - \alpha) + (\alpha^2 - \alpha)\eta(X)[\mu(\xi)(n-1) + \upsilon(\xi)] = 0. \tag{5.28}$$

So in view of (5.25), we get finally

$$\rho(X) = -\rho(\xi)\eta(X). \tag{5.29}$$

Since $(\tilde{\nabla}_\xi \tilde{S})(Y, \xi) = 0$, then from (5.2), we get

$$[\rho(\xi) + \mu(\xi)]\eta(X) = \upsilon(X), \tag{5.30}$$

that is

$$-\upsilon(\xi)\eta(X) = \upsilon(X). \tag{5.31}$$

Thus replacing Y with X in (5.27) and then summing of the equations (5.27), (5.29) and (5.31) we get

$$\rho(X) + \mu(X) + \upsilon(X) = -\eta(X)[\rho(\xi) + \mu(\xi) + \upsilon(\xi)]. \tag{5.32}$$

From the equation (5.25), it is clear that

$$\rho(X) + \mu(X) + \upsilon(X) = 0 \tag{5.33}$$

for any vector field X holds on M, which means that

$$\rho + \mu + \upsilon = 0.$$

Hence our proof is completed.

6 On semi-generalized recurrent Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold is called a semi-generalized recurrent manifold with respect to quarter-symmetric metric connection if its curvature tensor $\tilde{R}$ satisfies the condition

$$(\tilde{\nabla}_X \tilde{R})(Y, Z)W = \alpha_1(X)\tilde{R}(Y, Z)W + \beta_1(X)g(Z, W)Y, \tag{6.1}$$

where α_1 and β_1 defined as (1.5) for any vector field and $\tilde{\nabla}$ denotes the operator of covarient differentiation with respect to the metric.

Taking $Y = W = \xi$ in (6.1), we have

$$(\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = \alpha_1(X)\tilde{R}(\xi, Z)\xi + \beta_1(X)g(Z, \xi)\xi. \tag{6.2}$$

From (4.5), the left hand side of (6.2) can be written in the form

$$(\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = X\tilde{R}(\xi, Z)\xi - \tilde{R}(\tilde{\nabla}_X \xi, Z) - \tilde{R}(\xi, \tilde{\nabla}_X Z) - \tilde{R}(\xi, Z)\tilde{\nabla}_X \xi. \tag{6.3}$$

Now using (2.6), (2.16), (3.4), (3.6) and (3.11), the right hand site of the equation (6.3) becomes

$$(\tilde{\nabla}_X\tilde{R})(\xi, Z)\xi = -(\alpha^2-\alpha)(\alpha-1)\eta(Z)\phi X - (\alpha^2-\alpha)\eta(Z)\phi X. \tag{6.4}$$

Now using (3.11), the right hand side of (6.2) can be written in the form

$$(\tilde{\nabla}_X\tilde{R})(\xi, Z)\xi = \alpha_1(X)(\alpha^2-\alpha)[Z+\eta(Z)\xi] + \beta_1(X)\eta(Z)\xi. \tag{6.5}$$

Now from (6.4) and (6.5), we have

$$\begin{aligned} -(\alpha^2-\alpha)(\alpha\, -\, 1)\eta(Z)\phi X - (\alpha^2-\alpha)\eta(Z)\phi X \\ = \alpha_1(X)(\alpha^2-\alpha)[Z+\eta(Z)\xi] \\ + \beta_1(X)\eta(Z)\xi. \end{aligned} \tag{6.6}$$

Now putting $Z = \xi$ in (6.6), we obtain

$$(\alpha^2-\alpha)\tilde{\nabla}_X\xi + \alpha\tilde{\nabla}_X\xi = -\beta_1(X)\xi, \tag{6.7}$$

that is

$$\alpha^2\tilde{\nabla}_X\xi = -\beta_1(X)\xi. \tag{6.8}$$

Hence we can state the following theorem:

Theorem 6.1. *If a semi-generalized recurrent Lorentzian α-Sasakian manifold admits quarter-symmetric metric connection, the associated vector field ξ is not constant and $\nabla_X\xi$ is parallel to ξ, provided $\alpha \neq 0$.*

Permutting equation (6.1) with respect to X, Y, Z and adding the three equations and using Bianchi identity, we have

$$\begin{aligned} \alpha_1(X)\tilde{R}(Y,Z)W \,+\, \beta_1(X)g(Z,W)Y + \alpha_1(Y)\tilde{R}(Z,X)W + \beta_1(Y)g(X,W)Z \\ + \,\alpha_1(Z)\tilde{R}(X,Y)W + \beta_1(Z)g(Y,W)X = 0. \end{aligned} \tag{6.9}$$

Contracting (6.9) with respect to Y, we get

$$\begin{aligned} \alpha_1(X)\tilde{S}(Z,W) \,+\, n\beta_1(X)g(Z,W) + \tilde{R}'(Z,X,W,A) + \beta_1(Z)g(X,W) \\ -\, \alpha_1(Z)\tilde{S}(X,W) + \beta_1(Z)g(X,W) = 0. \end{aligned} \tag{6.10}$$

In view of $\tilde{S}(Z,W) = g(\tilde{Q}Z,W)$, the equation (6.10) becomes

$$\begin{aligned} \alpha_1(X)g(\tilde{Q}Z,W) \,+\, n\beta_1(X)g(Z,W) - g(\tilde{R}(Z,X)A,W) + \beta_1(Z)g(X,W) \\ -\, \alpha_1(Z)g(\tilde{Q}X,W) + \beta_1(Z)g(X,W) = 0. \end{aligned} \tag{6.11}$$

From (6.11), we have

$$\begin{aligned} \alpha_1(X)\tilde{Q}Z \,+\, n\beta_1(X)Z - \tilde{R}(Z,X)A + \beta_1(Z)X \\ -\, \alpha_1(Z)\tilde{Q}X + \beta_1(Z)X = 0. \end{aligned} \tag{6.12}$$

Contracting (6.12) with respect to Z, we obtain

$$\alpha_1(X)\tilde{r} + (n^2+2)\beta_1(X) - 2\tilde{S}(X,A) = 0. \tag{6.13}$$

Putting $X=\xi$ in (6.13), we get

$$\eta(A)\tilde{r} + (n^2+2)\eta(B) - 2(n-1)(\alpha^2-\alpha)\eta(A) = 0, \tag{6.14}$$

that is

$$\tilde{r} = \frac{1}{\eta(A)}[2(n-1)(\alpha^2-\alpha)\eta(A) - (n^2+2)\eta(B)], \tag{6.15}$$

where $\tilde{r}$ is the scalar curvature with respect to quarter-symmetric metric connection.
Hence we can state the following theorem:

Theorem 6.2. *The scalar curvature of a semi-generalized recurrent Lorentzian α-Sasakian manifold admitting a quarter-symmetric metric connection is related in terms of contact forms $\eta(A)$ and $\eta(B)$ as given by* (6.15).

7 On semi-generalized Ricci-recurrent Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold is called a semi-generalized Ricci-recurrent manifold with respect to quarter-symmetric metric connection if its Ricci tensor S satisfies the condition

$$(\tilde{\nabla}_X\tilde{S})(Y,Z) = \alpha_1(X)\tilde{S}(Y,Z) + n\beta_1(X)g(Y,Z), \tag{7.1}$$

where α_1 and β_1 defined as (1.5).

Taking $Z=\xi$ in (7.1), we have

$$(\tilde{\nabla}_X\tilde{S})(Y,\xi) = \alpha_1(X)\tilde{S}(Y,\xi) + n\beta_1(X)g(Y,\xi). \tag{7.2}$$

The left hand side of (7.2), clearly can be written in the form

$$(\tilde{\nabla}_X\tilde{S})(Y,\xi) = X\tilde{S}(Y,\xi) - \tilde{S}(\tilde{\nabla}_XY,\xi) - \tilde{S}(Y,\tilde{\nabla}_X\xi). \tag{7.3}$$

Using (3.2) and (3.7), the right hand site of the equation (7.3) becomes

$$-\tilde{S}(Y,\tilde{\nabla}_X\xi) + (n-1)\alpha(\alpha^2-\alpha)g(\phi X,Y). \tag{7.4}$$

The right hand site of (7.2) can be written as using (3.7)

$$\alpha_1(X)(n-1)(\alpha^2-\alpha)\eta(Y) + n\beta_1(X)\eta(Y). \tag{7.5}$$

From (7.4) and (7.5), we get

$$\tilde{S}(Y, \tilde{\nabla}_X \xi) + (n-1)\alpha(\alpha^2 - \alpha)g(\phi X, Y) = \alpha_1(X)(n-1)(\alpha^2 - \alpha)\eta(Y) + n\beta_1(X)\eta(Y). \tag{7.6}$$

Now putting $Y = \xi$ in (7.6), we obtain

$$\alpha_1(X)(n-1)(\alpha^2 - \alpha) + n\beta_1(X) = 0, \tag{7.7}$$

that is

$$\alpha_1(X) = -\frac{n}{(n-1)(\alpha^2 - \alpha)}\beta_1(X). \tag{7.8}$$

This leads to the following theorem:

Theorem 7.1. *If a semi-generalized Rici-Recurrent Lorentzian α-Sasakian manifold admits a quarter-symmetric metric connection, then*

$$\alpha_1(X) = -\frac{n}{(n-1)(\alpha^2 - \alpha)}\beta_1(X)$$

holds, that is, the 1-form α_1 and β_1 are in opposite direction.

A Lorentzian α-Sasakian manifold (M^n, g) with respect to quarter-symmetric metric connection is said to be an Einstein manifold if its Ricci tensor $\tilde{S}$ is of the form

$$\tilde{S}(X, Y) = kg(X, Y), \tag{7.9}$$

where k is constant. For an Einstein manifold,

$$(\tilde{\nabla}_U \tilde{S}) = 0$$

$\forall\ U \in \chi(M)$. From (7.1), we have

$$[k\alpha_1(X) + n\beta_1(X)]g(Y, Z) + [k\alpha_1(y) + n\beta_1(y)]g(Z, X) + [k\alpha_1(Z) + n\beta_1(Z)]g(X, Y) = 0. \tag{7.10}$$

Putting $Y = \xi$ in (7.10) and using (1.5) and (2.4), we obtain

$$[k\alpha_1(X) + n\beta_1(X)]\eta(Y) + [k\alpha_1(y) + n\beta_1(y)]\eta(X) + [k\alpha_1(Z) + n\beta_1(Z)]g(X, Y) = 0. \tag{7.11}$$

Now putting $X = Y = \xi$ in (7.11) and using (1.5), (2.2) and (2.4), we obtain

$$k\eta(A) + n\eta(B) = 0, \tag{7.12}$$

that is

$$\eta(A) = -\frac{n}{k}\eta(B). \tag{7.13}$$

Using (1.5) and (2.4) in the above relation, we have

$$\alpha_1(\xi) = -\frac{n}{k}\beta_1(\xi). \tag{7.14}$$

So, we have the following theorem:

Theorem 7.2. *If a semi-generalized Ricci-recurrent Lorentzian α-Sasakian manifold M admitting a quarter-symmetric metric connection is an Einstein manifold, then the contact form η(A) and η(B) and the 1-form α_1 and β_1 are both in opposite direction.*

8 Example of 3-dimensional Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

We consider a 3-dimensional manifold $M = \{(x, y, u) \in R^3\}$, where (x, y, u) are the standard coordinates of R^3. Let e_1, e_2, e_3 be the vector fields on M^3 given by

$$e_1 = e^{-u}\frac{\partial}{\partial x}, \quad e_2 = e^{-u}\frac{\partial}{\partial y}, \quad e_3 = e^{-u}\frac{\partial}{\partial u}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$

$$g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1, 1)$ tensor field ϕ is defined by

$$\phi e_1 = e_1, \quad \phi e_2 = e_2, \quad \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\eta(e_3) = -1,$$

$$\phi^2 X = X + \eta(X)e_3$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.
Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 e^{-u}, \quad [e_2, e_3] = e_2 e^{-u}.$$

Koszul's formula is defined by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Then from above formula we can calculate the followings:

$$\nabla_{e_1} e_1 = e_3 e^{-u}, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1 e^{-u},$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3 e^{-u}, \quad \nabla_{e_2} e_3 = e_2 e^{-u},$$

$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \alpha\phi X$ for $\alpha = e^{-u}$.
Hence the structure (ϕ, ξ, η, g) is a Lorentzian α-Sasakian manifold.
Using (2.16), we find $\tilde{\nabla}$, the quarter-symmetric metric connection on M following:

$$\tilde{\nabla}_{e_1} e_1 = e_3 e^{-u}, \ \tilde{\nabla}_{e_1} e_2 = 0, \ \tilde{\nabla}_{e_1} e_3 = e_1(e^{-u} - 1),$$

$$\tilde{\nabla}_{e_2} e_1 = 0, \quad \tilde{\nabla}_{e_2} e_2 = e_3 e^{-u}, \quad \tilde{\nabla}_{e_2} e_3 = e_2(e^{-u} - 1),$$

$$\tilde{\nabla}_{e_3} e_1 = 0, \quad \tilde{\nabla}_{e_3} e_2 = 0, \quad \tilde{\nabla}_{e_3} e_3 = 0.$$

Using (1.2), the torson tensor T, with respect to quarter-symmetric metric connection $\tilde{\nabla}$ as follows:

$$\tilde{T}(e_i, e_i) = 0, \quad \forall i = 1, 2, 3,$$

$$\tilde{T}(e_1, e_2) = 0, \quad \tilde{T}(e_1, e_3) = -e_1, \quad \tilde{T}(e_2, e_3) = -e_2.$$

Also,

$$(\tilde{\nabla}_{e_1} g)(e_2, e_3) = 0, \quad (\tilde{\nabla}_{e_2} g)(e_3, e_1) = 0, \quad (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0.$$

Thus M is Lorentzian α-Sasakian manifold with quarter-symmetric metric connection $\tilde{\nabla}$.

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