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Existence Results for a Fractional Boundary Value Problem via Critical Point Theory

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Abstract

In this paper, we consider the following boundary value problem

$$\begin{cases} D^{\alpha}_{T^-}(D^{\alpha}_{0^+}(D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(t)))) = f(t,u(t)), \quad t \in [0,T], \\ u(0) = u(T) = 0 \\ D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(0)) = D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(T)) = 0, \end{cases}$$

where $0 < \alpha \leq 1$ and $f\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is a continuous function, $D^{\alpha}_{0^+}$, $D^{\alpha}_{T^-}$ are respectively the left and right fractional Riemann–Liouville derivatives and we prove the existence of at least one solution for this problem.

Key words: Existence results, fractional differential equation, boundary value problem, critical point theory, minimization principle, Mountain pass theorem.

Third order, nonlinear differential equation, uniform stability, uniform ultimate boundedness, periodic solutions.

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1 Introduction

There are many papers that have studied differential equations of fractional order using fixed point theory, but few of them have studied these equations by using variational methods. In [2], F. Jiao and Y. Zhou are the first who have studied the following problem using variational techniques. They consider the problem

$$\begin{cases} \frac{d}{dt}\Big(\frac{1}{2}I_{0^+}^{\beta}(u'(t)) + \frac{1}{2}I_{T^-}^{\beta}(u'(t))\Big) + f(t,u(t)) = 0, & \text{p.p } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $I_{0^+}^{\beta}$ and $I_{T^-}^{\beta}$ are respectively the left and right fractional Riemann–Liouville integrals of order $0 \le \beta < 1$, $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ is a function which satisfies certain assumptions and they prove existence of solutions by using Minimization principle and Mountain Pass theorem.

In paper [1], the same authors used the same theory to study the following boundary value problem of fractional order

$$\begin{cases} D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t)) = f(t,u(t)), & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $D_{T^-}^{\alpha}$ and $D_{0^+}^{\alpha}$ are the left and right Riemann–Liouville fractional derivatives of order $0 < \alpha \le 1$ respectively and they prove existence of solutions by using the same techniques.

C. Bai used variational theory in [3] and he proved existence of an infinitely many solutions for the following perturbed nonlinear fractional boundary value problem depending on two parameters

$$\begin{cases} D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t)) = \lambda a(t)f(u(t)) + \nu g(t,u(t)), & \text{a.e.} t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

by using Ricceri's critical point theorem, where $0 < \alpha \le 1$, λ and ν are non-negative parameters, $a\colon [0,T]\to\mathbb{R}$, $f\colon\mathbb{R}\to\mathbb{R}$ and $g\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ are three given continuous functions.

The aim of this paper is to study the following fractional boundary value problem

$$\begin{cases} D_{T^-}^{\alpha}(D_{0^+}^{\alpha}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t)))) = f(t,u(t)), & t \in [0,T], \\ u(0) = u(T) = 0 \\ D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(0)) = D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(T)) = 0, \end{cases} \tag{1.1}$$

where $0 < \alpha \le 1$ and $f\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

We will work in a suitable Banach space, prove a compact embedding and a Poincaré inequality of fractional type. We will use an appropriate functional to find critical points that are solutions of our fractional boundary value problem and we will use minimization principle and the Mountain Pass theorem.

2 Preliminaries

In this section, we introduce some necessary definitions and properties of the fractional calculus which are used in this paper and can be found in ([4], [7], [11]).

Definition 2.1 Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha > 0$ for a function u denoted by $I^{\alpha}_{a^+}u$ and $I^{\alpha}_{b^-}u$, respectively, are defined by

$$I^{\alpha}_{a^+}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \quad t > a,$$

and

$$I^{\alpha}_{b^-}u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)ds, \quad t < b,$$

provided that the right-hand side is pointwise defined on $[a,b]$, where $\Gamma(\alpha)$ is the gamma function.

Definition 2.2 Let u be a function defined on $[a,b]$. For $n-1 \leq \alpha < n$ ($n \in \mathbb{N}^*$), the left and right Riemann–Liouville fractional derivatives of order α for a function u denoted by $D^{\alpha}_{a^+}u$ and $D^{\alpha}_{b^-}u$, respectively, are defined by

$$D^{\alpha}_{a^+}u(t) = \frac{d^n}{dt^n}I^{n-\alpha}_{a^+}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (t-s)^{n-\alpha-1}u(t)\,ds, \quad t > a,$$

and

$$D^{\alpha}_{b^-}u(t) = (-1)^n\frac{d^n}{dt^n}I^{n-\alpha}_{b^-}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (s-t)^{n-\alpha-1}u(t)\,ds, \quad t < b,$$

provided that the right-hand side is pointwise defined.

In particular, for $\alpha = n$, $D^{\alpha}_{a^+}u(t) = D^n u(t)$ and $D^{\alpha}_{b^-}u(t) = (-1)^n D^n u(t)$, $t \in [a,b]$.

Definition 2.3 The left and right Caputo fractional derivatives of order α for a function $u \in AC^n([a,b],\mathbb{R})$ is defined by

$${}^C D^{\alpha}_{a^+}u(t) = I^{n-\alpha}_{a^+}D^n u(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}u^{(n)}(s)\,ds, \quad t > a,$$

and

$${}^c D^{\alpha}_{b^-}u(t) = (-1)^n I^{n-\alpha}_{b^-}D^n u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (s-t)^{n-\alpha-1}u^{(n)}(s)\,ds, \quad t < b.$$

Proposition 2.4 ([7], [8]) *If $D^{\alpha}_{a^+}u \in L^1([a,b])$ and $n-1 \leq \alpha < n$, then*

$$I^{\alpha}_{a^+}D^{\alpha}_{a^+}u(t) = u(t) + \sum_{j=1}^{n} c_j(t-a)^{\alpha-j}$$

with $c_j = \frac{D^{\alpha-j}_{a^+}u(a)}{\Gamma(\alpha-j+1)} \in \mathbb{R}$, $j = 1,2,\ldots,n$ *and*

$$I^{\alpha}_{b^-}D^{\alpha}_{b^-}u(t) = u(t) + \sum_{j=1}^{n} c'_j(b-t)^{\alpha-j}$$

with $c'_j = \frac{(-1)^{n-j}D^{\alpha-j}_{b^-}u(b)}{\Gamma(\alpha-j+1)} \in \mathbb{R}$, $j = 1,2,\ldots,n$.

Proposition 2.5 ([4], [7], [11]) *Suppose that ${}^cD^\alpha_{a^+}u$ and $D^\alpha_{a^+}u$ exist. Then*

$$ {}^cD^\alpha_{a^+}u(t) = D^\alpha_{a^+}u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, \quad t \in [a,b], $$

and

$$ {}^cD^\alpha_{b^-}u(t) = D^\alpha_{b^-}u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}, \quad t \in [a,b]. $$

In particular, if $0 < \alpha < 1$, we have

$$ {}^cD^\alpha_{a^+}u(t) = D^\alpha_{a^+}u(t) - \frac{u(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad t \in [a,b], $$

and

$$ {}^cD^\alpha_{b^-}u(t) = D^\alpha_{b^-}u(t) - \frac{u(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}, \quad t \in [a,b]. $$

We state now notations and techniques which are used in what follows.

Theorem 2.6 ([5], [9]) *Let $\varphi \colon U \to \mathbb{R}$ be a continuous convex functional on a Banach space X. Then φ is weakly lower semi-continuous, i.e. for every sequence $(u_n)_{n\in\mathbb{N}}$ converging weakly to $u \in X$, we have*

$$ \varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n). $$

Definition 2.7 Let $\varphi \in C^1(X,\mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \to 0$ when $n \to +\infty$ in X', the dual space of X, possesses a convergent subsequence, then we say that φ satisfies the Palais–Smale condition (denoted by (P.S) condition).

Theorem 2.8 (Minimization principle [5], [10]) *Let X be a real reflexive Banach space. If the functional $\varphi \colon X \to \mathbb{R}$ is weakly lower semi-continuous and coercive, i.e. $\lim_{\|u\|\to+\infty} \varphi(u) = +\infty$, then there exists $u_0 \in X$ such that $\varphi(u_0) = \inf_{u\in X} \varphi(u)$. Moreover, if φ is also Gateaux differentiable on X, then $\varphi'(u_0) = 0$.*

Theorem 2.9 (Mountain pass theorem [5], [9]) *Let X be a real Banach space and $\varphi \in C^1(X,\mathbb{R})$ satisfying the (P.S) condition. Suppose that*

(i) $\varphi(0) = 0$;

(ii) there exist $\rho > 0$ and $\sigma > 0$ such $\varphi(z) \geq \sigma$ for all $z \in X$ with $\|z\| = \rho$;

(iii) there exists z_1 in X with $\|z_1\| > \rho$ such that $\varphi(z_1) \leq \sigma$.

Then φ possesses a critical value such that $c \geq \sigma$. Moreover, c can be characterized as

$$ c = \inf_{g\in\Lambda} \max_{z\in g([0,1])} \varphi(z), $$

where

$$ \Lambda = \big\{ g \in C([0,1],X) : g(0) = 0,\ g(1) = z_1 \big\}. $$

3 Variational structure

To establish a variational structure for the boundary value problem (1.1), it is necessary to construct appropriate function spaces.

In this section, we set up a variational structure that allows us to reduce the existence of solutions for the problem (1.1) to find critical points of the corresponding functional defined on the following space $E_0^{2\alpha,p}$ with $p = 2$ and $0 < \alpha \leq 1$ denoted by $E_0^{2\alpha}$.

Definition 3.1 Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{\alpha,p} = \Big(\|u\|_{L^p}^p + \|D_{0^+}^\alpha u\|_{L^p}^p\Big)^{\frac{1}{p}}, \quad \forall u \in E_0^{\alpha,p} \tag{3.1}$$

with $\|u\|_{L^p} = (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}$.

We put

$$E_0^{\alpha,p} = \Big\{u \in L^p([0,T]);\ D_{0^+}^\alpha u \in L^p([0,T]);\ u(0) = u(T) = 0\Big\}.$$

Definition 3.2 The fractional derivative space $E_0^{2\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R})$ with respect to the norm

$$\|u\|_{2\alpha,p} = \Big(\|u\|_{L^p}^p + \|D_{0^+}^\alpha u\|_{L^p}^p + \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p}^p\Big)^{\frac{1}{p}}, \quad \forall u \in E_0^{2\alpha,p}. \tag{3.2}$$

Remark 3.3 It is obvious that the fractional derivative space $E_0^{2\alpha,p}$ is the space of functions $u \in L^p([0,T],\mathbb{R})$ having an α-order Riemann–Liouville fractional derivative $D_{0^+}^\alpha u \in L^p([0,T],\mathbb{R})$, $D_{T^-}^\alpha (D_{0^+}^\alpha u) \in L^p([0,T],\mathbb{R})$, $u(0) = u(T) = 0$ and $D_{T^-}^\alpha (D_{0^+}^\alpha u(0)) = D_{T^-}^\alpha (D_{0^+}^\alpha u(T)) = 0$. We put

$$\begin{aligned} E_0^{2\alpha,p} = \big\{&u \in L^p([0,T]);\ D_{0^+}^\alpha u \in L^p([0,T]) \text{ and } D_{T^-}^\alpha (D_{0^+}^\alpha u) \in L^p([0,T],\mathbb{R}),\\ &u(0) = u(T) = 0,\ D_{T^-}^\alpha (D_{0^+}^\alpha u(0)) = D_{T^-}^\alpha (D_{0^+}^\alpha u(T)) = 0\big\}. \end{aligned}$$

Lemma 3.4 ([1], [4], [7]) *For any $u \in L^p([0,T],\mathbb{R})$, we have*

$$\|I_{0^+}^\alpha u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p}, \tag{3.3}$$

and

$$\|I_{T^-}^\alpha u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p}. \tag{3.4}$$

Proposition 3.5 *For all $u \in E_0^{2\alpha,p}$, we have*

$$I_{0^+}^\alpha D_{0^+}^\alpha u = u \quad \textit{and} \quad I_{T^-}^\alpha D_{T^-}^\alpha u = u.$$

Proof The proof is easy by using Proposition 2.4. $\square$

Proposition 3.6 [1] *For any $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{0^+}^\alpha u\|_{L^p}. \tag{3.5}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|D_{0^+}^\alpha u\|_{L^p}, \tag{3.6}$$

where $\|u\|_\infty = \sup_{t \in [0,T]} |u(t)|$.

Proposition 3.7 *For any $u \in E_0^{2\alpha,p}$, we have*

$$\|D_{0^+}^\alpha u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p}. \tag{3.7}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \cdot \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p}. \tag{3.8}$$

Proof By Proposition 2.4, we have

$$I_{T^-}^\alpha D_{T^-}^\alpha u(t) = u(t) - \frac{D_{T^-}^{\alpha-1} u(T)}{\Gamma(\alpha)} (T-t)^{\alpha-1}.$$

For all $u \in E_0^{2\alpha,p}$, we obtain that $I_{b^-}^\alpha D_{b^-}^\alpha u(t) = u(t)$ and by (3.4), we have

$$\|I_{T^-}^\alpha (D_{T^-}^\alpha (D_{0^+}^\alpha u))\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p}$$

which implies that

$$\|D_{0^+}^\alpha u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p}.$$

We have also by (3.6), (3.5) that

$$\begin{aligned} \|u\|_\infty &\leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|D_{0^+}^\alpha u\|_{L^p} \\ &\leq \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \cdot \frac{T^\alpha}{\Gamma(\alpha+1)} \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p}. \end{aligned}$$

□

Proposition 3.8 *There exist $\gamma_1, \gamma_2 \in \mathbb{R}_*^+$ such that*

$$\gamma_1 \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p} \leq \|u\|_{2\alpha,p} \leq \gamma_2 \|D_{T^-}^\alpha (D_{0^+}^\alpha u)\|_{L^p}.$$

Proof Indeed, we have

$$\begin{aligned}
\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p} &\leq \|u\|_{2\alpha,p} \leq c_1\Big(\|u\|_{L^p} + \|D^{\alpha}_{0^+}u\|_{L^p} + \|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}\Big)\\
&\leq c_1\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{0^+}u\|_{L^p} + \|D^{\alpha}_{0^+}u\|_{L^p} + \|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}\Big]\\
&\leq c_1\Big[\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + 1\Big)\|D^{\alpha}_{0^+}u\|_{L^p} + \|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}\Big]\\
&\leq c_1\Big[\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + 1\Big)\frac{T^{\alpha}}{\Gamma(\alpha+1)}\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p} + \|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p}\Big]\\
&\leq c_1\Big[\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + 1\Big)\frac{T^{\alpha}}{\Gamma(\alpha+1)} + 1\Big]\|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p},
\end{aligned}$$

for some $c_1 > 0$, thus, the two norms are equivalent with

$$\gamma_1 = 1 \quad \text{and} \quad \gamma_2 = c_1\Big[\Big(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + 1\Big)\frac{T^{\alpha}}{\Gamma(\alpha+1)} + 1\Big].$$

□

Remark 3.9 We can endowed $E_0^{2\alpha,p}$ with respect to the equivalent norm

$$\|u\|_{2\alpha,p} = \|D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u)\|_{L^p} = \Big(\int_0^T |D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(t))|^p dt\Big)^{\frac{1}{p}}. \tag{3.9}$$

Proposition 3.10 ([2], [4]) *If $u \in E_0^{\alpha,p}$ and $v \in C_0^{\infty}([0,T])$, then*

$$\int_0^T D^{\alpha}_{0^+}u(t)v(t)dt = \int_0^T u(t)D^{\alpha}_{T^-}v(t)dt.$$

Proposition 3.11 *If $u \in E^{2\alpha,p}$ and $v \in C_0^{\infty}([0,T])$, then*

$$\int_0^T D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(t))v(t)dt = \int_0^T u(t)D^{\alpha}_{T^-}(D^{\alpha}_{0^+}v(t))dt.$$

Proof We have, by using integration by parts and Dirichlet formula (see [7]),

$$\begin{aligned}
\int_0^T v(t)D^{\alpha}_{T^-}(D^{\alpha}_{0^+}u(t))\,dt &= \frac{-1}{\Gamma(1-\alpha)}\int_0^T \frac{d}{dt}\Big[\int_t^T (s-t)^{-\alpha}D^{\alpha}_{0^+}u(s)\,ds\Big]v(t)\,dt\\
&= \Big[\frac{-1}{\Gamma(1-\alpha)}v(t)\int_t^T (s-t)^{-\alpha}D^{\alpha}_{0^+}u(s)\,ds\Big]_{t=0}^{t=T}\\
&\quad + \frac{1}{\Gamma(1-\alpha)}\int_0^T \Big[\int_t^T (s-t)^{-\alpha}D^{\alpha}_{0^+}u(s)\,ds\Big]v'(t)\,dt
\end{aligned}$$

$$= \frac{v(0)}{\Gamma(1-\alpha)} \int_0^T (s-0)^{-\alpha} D^\alpha_{0+} u(s)\, ds + \frac{1}{\Gamma(1-\alpha)} \int_0^T \Big[\int_0^s (s-t)^{-\alpha} v'(t)\, dt \Big] D^\alpha_{0+} u(s)\, ds$$

$$= \frac{v(0)}{\Gamma(1-\alpha)} \int_0^T (s-0)^{-\alpha} D^\alpha_{0+} u(s)\, ds + \int_0^T {}^c D^\alpha_{0+} v(s) D^\alpha_{0+} u(s)\, ds$$

$$= \frac{v(0)}{\Gamma(1-\alpha)} \int_0^T (s-0)^{-\alpha} D^\alpha_{0+} u(s)\, ds + \int_0^T \Big[D^\alpha_{0+} v(s) - \frac{v(0)}{\Gamma(1-\alpha)} (s-0)^{-\alpha} \Big] D^\alpha_{0+} u(s)\, ds$$

$$= \frac{v(0)}{\Gamma(1-\alpha)} \int_0^T (s-0)^{-\alpha} D^\alpha_{0+} u(s)\, ds$$

$$+ \int_0^T D^\alpha_{0+} u(s) D^\alpha_{0+} v(s)\, ds - \frac{v(0)}{\Gamma(1-\alpha)} \int_0^T (s-0)^{-\alpha} D^\alpha_{0+} u(s)\, ds$$

$$= \int_0^T D^\alpha_{0+} u(s) D^\alpha_{0+} v(s)\, ds.$$

By using integration by parts for the second time in the last term, we obtain,

$$\int_0^T D^\alpha_{0+} u(t) D^\alpha_{0+} v(t)\, dt = \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{d}{dt} \Big[\int_0^t (t-s)^{-\alpha} D^\alpha_{0+} u(s)\, ds \Big] D^\alpha_{0+} v(t)\, dt$$

$$= \Big[\frac{1}{\Gamma(1-\alpha)} D^\alpha_{0+} v(t) \int_0^t (t-s)^{-\alpha} D^\alpha_{0+} u(s)\, ds \Big]_{t=0}^{t=T}$$

$$- \frac{1}{\Gamma(1-\alpha)} \int_0^T \Big[\int_0^t (t-s)^{-\alpha} u(s) ds \Big] D^{\alpha+1}_{0+} v(t)\, dt$$

$$= \frac{D^\alpha_{0+} v(T)}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} D^\alpha_{0+} u(s)\, ds$$

$$- \frac{1}{\Gamma(1-\alpha)} \int_0^T \Big[\int_s^T (t-s)^{-\alpha} D^{\alpha+1}_{0+} v(t) dt \Big] u(s)\, ds$$

$$= \frac{D^\alpha_{0+} v(T)}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} D^\alpha_{0+} u(s)\, ds$$

$$- \frac{1}{\Gamma(1-\alpha)} \int_0^T \Big[\int_s^T (t-s)^{-\alpha} (D^\alpha_{0+} v)'(t) dt \Big] u(s)\, ds$$

$$= \frac{D^\alpha_{0+} v(T)}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} D^\alpha_{0+} u(s)\, ds + \int_0^T {}^c D^\alpha_{T-} (D^\alpha_{0+} v(s)) u(s)\, ds$$

$$= \frac{D^\alpha_{0+} v(T)}{\Gamma(1-\alpha)} \int_0^T (T-s)^{-\alpha} D^\alpha_{0+} u(s)\, ds$$

$$+ \int_0^T \Big[D^\alpha_{T-} (D^\alpha_{0+} v(s)) - \frac{D^\alpha_{0+} v(T)}{\Gamma(1-\alpha)} (T-s)^{-\alpha} \Big] u(s)\, ds$$

$$= \int_0^T D^\alpha_{T-} (D^\alpha_{0+} v(s)) u(s)\, ds.$$

$\square$

Proposition 3.12 $E_0^{2\alpha,p}$ *is a Banach space.*

Proof Let $(u_n)_{n\geq 1}$ be a Cauchy sequence in $E_0^{2\alpha,p}$. Then $(u_n)_{n\geq 1}$, $(D_{0+}^{\alpha}u_n)_{n\geq 1}$ and $(D_{T-}^{\alpha}(D_{0+}^{\alpha}u_n))_{n\geq 1}$ are Cauchy sequences in $L^p([0,T])$. In fact, by (3.2) we have $\|u_n - u_m\|_{2\alpha,p} \to 0$ as $n, m \to +\infty$ which implies that $\|u_n - u_m\|_{L^p} \to 0$, $\|D_{0+}^{\alpha}u_n - D_{0+}^{\alpha}u_m\|_{L^p} \to 0$ and $\|D_{T-}^{\alpha}(D_{0+}^{\alpha}u_n) - D_{T-}^{\alpha}(D_{0+}^{\alpha}u_m)\|_{L^p} \to 0$ as $n, m \to +\infty$.

Since $L^p([0,T])$ is a Banach space, there exist functions u_1, u_2, u_3 in $L^p([0,T])$, such that $u_n \to u_1$, $D_{0+}^{\alpha}u_n \to u_2$ and $D_{T-}^{\alpha}(D_{0+}^{\alpha}u_n) \to u_3$ in $L^p([0,T])$ as $n \to +\infty$. We now show that $D_{0+}^{\alpha}u = u_2$ and $D_{T-}^{\alpha}(D_{0+}^{\alpha}u) = u_3$. In fact, by the Proposition 3.10, we have

$$\int_0^T D_{0+}^{\alpha}u_n(t)v(t)dt = \int_0^T u_n(t)D_{T-}^{\alpha}v(t)\,dt, \quad \forall v \in C_0^{\infty}([0,T]),$$

and then by passing to the limit and using the Lebesgue's dominated convergence theorem, we obtain that

$$\int_0^T u_2(t)v(t)dt = \int_0^T u_1(t)D_{T-}^{\alpha}v(t)\,dt, \quad \forall v \in C_0^{\infty}([0,T]).$$

Therefore $D_{0+}^{\alpha}u_1 = u_2$.

We have also, by Proposition 3.11,

$$\int_0^T D_{T-}^{\alpha}(D_{0+}^{\alpha}u_n(t))v(t)\,dt = \int_0^T u_n(t)D_{T-}^{\alpha}(D_{0+}^{\alpha}v(t))\,dt, \quad \forall v \in C_0^{\infty}([0,T]),$$

and then by passing to the limit and using the Lebesgue's dominated convergence theorem, we obtain that

$$\int_0^T u_3(t)v(t)\,dt = \int_0^T u_1(t)D_{T-}^{\alpha}(D_{0+}^{\alpha}v(t))\,dt, \quad \forall v \in C_0^{\infty}([0,T]),$$

then $D_{T-}^{\alpha}(D_{0+}^{\alpha}u_1(t)) = u_3(t)$, i.e. $\lim_{n\to+\infty}\|u_n - u_1\|_{2\alpha,p} = 0$. Consequently $E_0^{2\alpha,p}$ is a Banach space. $\square$

Lemma 3.13 *The operator*

$$\begin{aligned} T\colon E_0^{2\alpha,p} &\to T(E_0^{2\alpha,p}) \subset L^p([0,T]) \times L^p([0,T]) \times L^p([0,T]) = L_3^p([0,T]) \\ u &\to T(u) = \big(u, D_{0+}^{\alpha}u, D_{T-}^{\alpha}(D_{0+}^{\alpha}u)\big) \end{aligned}$$

is an isometric isomorphic mapping.

Proof It is clear that T is a linear operator and we will show that T preserves the norms, i.e.

$$\forall u \in E_0^{2\alpha,p} : \|Tu\|_{L_3^p} = \|u\|_{E_0^{2\alpha,p}}.$$

Indeed, we have

$$\|(u, D_{0+}^{\alpha}u, D_{T-}^{\alpha}(D_{0+}^{\alpha}u))\|_{L_3^p} = \|u\|_{E_0^{2\alpha,p}}$$

which is equivalent to

$$\|u\|_{L^p} + \|D^{\alpha}_{0^+} u\|_{L^p} + \|D^{\alpha}_{T^-}(D^{\alpha}_{0^+} u)\|_{L^p} = \|u\|_{E_0^{2\alpha,p}}$$

and this is true by the definition of the equivalent norm of $E_0^{2\alpha,p}$. □

Proposition 3.14 $E_0^{2\alpha,p}$ *is a reflexive space.*

Proof Since, $L^p([0,T],\mathbb{R})$ is a reflexive Banach space, the cartesian product space

$$L^p_3([0,T],\mathbb{R}) = L^p([0,T],\mathbb{R}) \times L^p([0,T],\mathbb{R}) \times L^p([0,T],\mathbb{R})$$

is also a reflexive Banach space with respect to the norm

$$\|u\|_{L^p_3([0,T])} = \sum_{i=1}^{3} \Big(\|u_i\|^p_{L^p}\Big)^{\frac{1}{p}} \quad \text{where } u = (u_1, u_2, u_3) \in L^p_3([0,T],\mathbb{R}).$$

We have then

$$\begin{aligned} T\colon E_0^{2\alpha,p} &\to T(E_0^{2\alpha,p}) \subset L^p_3 \\ u &\to T(u) = \big(u,\, D^{\alpha}_{0^+} u,\, D^{\alpha}_{T^-}(D^{\alpha}_{0^+} u)\big). \end{aligned}$$

is an isometric isomorphic, then $T(E_0^{2\alpha,p})$ is a closed subspace of L^p_3 and by [6, Theorem 4.10.5] $T(E_0^{2\alpha,p})$ is reflexive. Consequensly $E_0^{2\alpha,p}$ is also reflexive (see [6, Lemma 4.10.4]). □

Proposition 3.15 $E_0^{2\alpha,p}$ *is a separable space.*

Proof Since, $L^p([0,T],\mathbb{R})$ is a separable Banach space, the cartesian space

$$L^p_3([0,T],\mathbb{R}) = L^p([0,T],\mathbb{R}) \times L^p([0,T],\mathbb{R}) \times L^p([0,T],\mathbb{R})$$

is also a separable Banach space with respect to the norm

$$\|u\|_{L^p_3([0,T])} = \sum_{i=1}^{3} \Big(\|u_i\|^p_{L^p}\Big)^{\frac{1}{p}} \quad \text{where } u = (u_1, u_2, u_3) \in L^p_3([0,T],\mathbb{R}).$$

Then, the space $T(E_0^{2\alpha,p}) \subset L^p_3$ is also separable (see [12, Proposition III.22]).

Moreover, the operator

$$\begin{aligned} T\colon E_0^{2\alpha,p} &\to T(E_0^{2\alpha,p}) \subset L^p_3 \\ u &\to T(u) = (u, D^{\alpha}_{0^+} u, D^{\alpha}_{T^-}(D^{\alpha}_{0^+} u)). \end{aligned}$$

is an isometric isomorphic. Consequently $E_0^{2\alpha,p}$ is a separable space. □

Proposition 3.16 [1] *The space $E_0^{\alpha,p}$ is compactly embedded in $C([0,T],\mathbb{R})$.*

Proposition 3.17 *The space $E_0^{2\alpha,p}$ is compactly embedded in $C([0,T],\mathbb{R})$.*

Proof By Proposition 3.16, the injection $i_1\colon E_0^{\alpha,p} \to C([0,T],\mathbb{R})$ is compact and the injection $i_2\colon E_0^{2\alpha,p} \to E_0^{\alpha,p}$ is obviously continuous. Then $i_2 \circ i_1$ is compact. □

4 Main results

Now, we give the definition of a solution of the boundary value problem (1.1).

Definition 4.1 A functional $u\colon [0,T] \to \mathbb{R}$ is called solution of the boundary value problem (1.1) if:

(i) $D_{T^-}^{\alpha-1}(D_{0^+}^{\alpha}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))))$, $(D_{0^+}^{\alpha-1}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))))$, $D_{T^-}^{\alpha-1}(D_{0^+}^{\alpha}u(t))$ and $D_{0^+}^{\alpha-1}u(t)$ are derivatives for all $t \in [0,T]$, and

(ii) u satisfies (1.1).

Proposition 4.2 *All critical point of the functional φ defined on $E_0^{2\alpha}$ by*

$$\varphi(u) = \frac{1}{2}\int_0^T |D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))|^2\,dt - \int_0^T F(t,u(t))\,dt$$

where $F(t,u) = \int_0^u f(t,s)\,ds$ is a solution of the boundary value problem (1.1).

Proof A weak solution of the problem (1.1) is a critical point of φ and satisfy $\varphi'(u) = 0$, i.e.

$$\begin{aligned}\varphi'(u)(v) = &\int_0^T D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))D_{T^-}^{\alpha}(D_{0^+}^{\alpha}v(t))\,dt\\ &-\int_0^T f(t,u(t))v(t)\,dt = 0, \quad \forall v \in E_0^{2\alpha}.\end{aligned}$$

In fact, for all $u, v \in E_0^{2\alpha}$, we obtain by Proposition 3.11,

$$\begin{aligned}&\int_0^T D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))D_{T^-}^{\alpha}(D_{0^+}^{\alpha}v(t))\,dt - \int_0^T f(t,u(t))v(t)dt = 0\\ \Leftrightarrow &\int_0^T D_{0^+}^{\alpha}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t)))D_{0^+}^{\alpha}v(t)\,dt - \int_0^T f(t,u(t))v(t)\,dt = 0\\ \Leftrightarrow &\int_0^T D_{T^-}^{\alpha}(D_{0^+}^{\alpha}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))))v(t)\,dt - \int_0^T f(t,u(t))v(t)\,dt = 0\\ \Leftrightarrow &\int_0^T \Big[D_{T^-}^{\alpha}(D_{0^+}^{\alpha}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t)))) - f(t,u(t))\Big]v(t)\,dt = 0\\ \Leftrightarrow &D_{T^-}^{\alpha}(D_{0^+}^{\alpha}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t)))) = f(t,u(t)) \quad \text{a.e. } t \in [0,T].\end{aligned}$$

Because f is continuous, then $D_{T^-}^{\alpha}(D_{0^+}^{\alpha}(D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t)))) = f(t,u(t))$ for all $t \in [0,T]$. □

We can prove easily the following lemma.

Lemma 4.3 *The functional $H\colon E_0^{2\alpha} \to \mathbb{R}$ defined by*

$$H(u) = \frac{1}{2}\|u\|_{2\alpha}^2$$

is convex and continuous on $E_0^{2\alpha}$.

Remark 4.4 The functional H is then weakly lower semicontinuous (see [5, Theorem 1.5.3]).

Lemma 4.5 *The functional* $G\colon E_0^{2\alpha} \to \mathbb{R}$ *defined by*

$$G(u) = \int_0^T F(t, u(t))\, dt$$

is weakly continuous on $E_0^{2\alpha}$.

Proof In fact, according to Proposition (3.17), if $u_n \rightharpoonup u$ in $E_0^{2\alpha}$, then $u_n \to u$ in $C([0,1], \mathbb{R})$. Therefore, since F is continuous with respect to the second variable for all $t \in [0,T]$, then $F(t, u_n(t)) \to F(t, u(t))$ for all $t \in [0,T]$. Since the sequence (u_n) converges in $C([0,T], \mathbb{R})$, so it is bounded in the same space, and therefore there exists $R > 0$ such that $|u_n(t)| \le \|u_n\|_\infty \le R$, and we obtain

$$|F(t, u_n(t))| \le \sup_{y \in [-R,R]} F(t,y), \quad \forall t \in [0,T].$$

By the Lebesgue's dominated convergence theorem, we have $\int_0^T F(t, u_n(t))\, dt \to \int_0^T F(t, u(t))\, dt$, which means that the functional $u \to \int_0^T F(t, u(t))\, dt$ is weakly continuous on $E_0^{2\alpha}$, and then the functional G is weakly lower semicontinuous. $\square$

Theorem 4.6 *Suppose that there exist constant* $\theta \in [0, 2)$ *and a function* $a(.) \in C([0,T], \mathbb{R})$ *with* $a^* = \sup_{t\in[0,T]} a(t) > 0$ *such that*

$$\limsup_{|u| \to +\infty} \frac{F(t,u)}{|u|^\theta} \le a(t),$$

uniformly with respect to $t \in [0,T]$ *hold, then the boundary value problem* (1.1) *has at least one solution.*

Proof By the hypothesis, there exists $R_1 > 0$ such that $F(t,x) \le a(t)|x|^\theta$ for all $t \in [0,T]$ and all $|x| \ge R_1$, which combined with the continuity of $F(t,x) - a(t)|x|^\theta$ on $[0,T] \times [-R_1, R_1]$, gives that there exists a constant $C_1 > 0$ such that

$$F(t,x) \le \tau_1(t)|x|^\theta + C_1 \quad \text{for all } t \in [0,T] \text{ and all } x \in \mathbb{R}. \tag{4.1}$$

For $u \in E_0^{2\alpha}$ and by the relations (3.6), (4.1) we obtain

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |D_{T^-}^\alpha (D_{0^+}^\alpha u(t))|^2 dt - \int_0^T F(t, u(t))\, dt \\
&\ge \frac{1}{2}\|u\|_{2\alpha}^2 - \int_0^T a(t)|u(t)|^\theta dt - \int_0^T C_1\, dt
\end{aligned}$$

$$\ge \frac{1}{2}\|u\|_{2\alpha}^2 - a^* \int_0^T |u(t)|^\theta\, dt - C_1 T \ge \frac{1}{2}\|u\|_{2\alpha}^2 - a^* \|u\|_\infty^\theta T - C_1 T$$

$$\ge \frac{1}{2}\|u\|_{2\alpha}^2 - a^* \Big(\frac{T^{\alpha - \frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}} \cdot \frac{T^\alpha}{\Gamma(\alpha+1)} \Big)^\theta T \|u\|_{2\alpha}^\theta - C_1 T.$$

Then, $\varphi(u) \to +\infty$ as $\|u\|_{2\alpha} \to +\infty$, and hence φ is coercive. By Lemma 4.5 and Remark 4.4, the functional φ is weakly lower semicontinuous. We conclude the proof of Theorem 4.6 by using Theorem 2.8. $\square$

Example 4.7 We consider the following problem

$$\begin{cases} D^{\frac{1}{4}}_{1^-}(D^{\frac{1}{4}}_{0^+}(D^{\frac{1}{4}}_{1^-}(D^{\frac{1}{4}}_{0^+}u(t)))) = f(t, u(t)), \quad t \in [0, 1], \\ u(0) = u(1) = 0, \\ D^{\frac{1}{4}}_{1^-}(D^{\frac{1}{4}}_{0^+}u(0)) = D^{\frac{1}{4}}_{1^-}(D^{\frac{1}{4}}_{0^+}u(T)) = 0, \end{cases} \tag{4.2}$$

with

$$f(t, y) = \begin{cases} ty^{\frac{1}{2}}, & \text{if } y \geq 0 \\ t(-y)^{\frac{1}{2}}, & \text{if } y \leq 0 \end{cases}$$

and

$$F(t, x) = \int_0^x f(t, y)\, dy = \frac{2}{3} t|x|^{\frac{3}{2}}.$$

The hypotheses of Theorem 4.6 hold with $\theta = \frac{3}{2}$ and $a(t) = \frac{2}{3}t$, where $a^* = \frac{2}{3} > 0$. In fact

$$\limsup_{|x| \to +\infty} \frac{F(t, x)}{|x|^{\frac{3}{2}}} = \frac{2}{3}t, \quad \text{uniformly with respect to } t \in [0, 1].$$

Thus problem (4.2) has at least one solution.

Theorem 4.8 *Suppose that there exists a function $b(.) \in C([0, T], \mathbb{R})$ with $b^* = \sup_{t \in [0,T]} b(t) > 0$ such that*

$$\limsup_{|x| \to +\infty} \frac{F(t, x)}{|x|^2} \leq b(t)$$

uniformly with respect to $t \in [0, T]$ with

$$\frac{1}{2} - b^* \cdot \frac{T^{5\alpha - 1}}{\Gamma^2(\alpha)\Gamma^2(\alpha + 1)(2(\alpha - 1) + 1)} > 0$$

hold, then the boundary value problem (1.1) *has at least one solution.*

Proof By the hypotheses, there exists $R_2 > 0$ such that

$$F(t, x) \leq b(t)|x|^2 \quad \text{for all } t \in [0, T] \text{ and all } |x| \geq R_2,$$

which combined with the continuity of $F(t, x) - b(t)|x|^2$ on $[0, T] \times [-R_2, R_2]$, gives that there exists a constant $C_2 > 0$ such that

$$F(t, x) \leq b(t)|x|^2 + C_2 \quad \text{for all } t \in [0, T] \text{ and all } x \in \mathbb{R}; \tag{4.3}$$

For $u \in E_0^{2\alpha}$ and by the relations (3.6), (4.3), we obtain

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |D_{T^-}^{\alpha}(D_{0^+}^{\alpha}u(t))|^2 dt - \int_0^T F(t,u(t))\,dt \\
&\geq \frac{1}{2}\|u\|_{2\alpha}^2 - \int_0^T b(t)|u(t)|^2 dt - \int_0^T C_2\,dt \\
&\geq \frac{1}{2}\|u\|_{2\alpha}^2 - b^*\int_0^T |u(t)|^2\,dt - C_2 T \\
&\geq \frac{1}{2}\|u\|_{2\alpha}^2 - b^*\|u\|_{\infty}^2 T - C_2 T \\
&\geq \frac{1}{2}\|u\|_{2\alpha}^2 - b^*\Big(\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}}\cdot\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big)^2 T\|u\|_{2\alpha}^2 - C_2 T \\
&\geq \frac{1}{2}\|u\|_{2\alpha}^2 - b^* T\cdot\frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2(\alpha-1)+1)}\cdot\frac{T^{2\alpha}}{\Gamma^2(\alpha+1)}\|u\|_{2\alpha}^2 - C_2 T \\
&\geq \Big(\frac{1}{2} - b^*\cdot\frac{T^{5\alpha-1}}{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)}\Big)\|u\|_{2\alpha}^2 - C_2 T.
\end{aligned}$$

Then, $\varphi(u) \to +\infty$ as $\|u\|_{2\alpha} \to +\infty$, and hence φ is coercive. By Lemma 4.5 and Remark 4.4, the functional φ is weakly lower semicontinuous. We conclude the proof of Theorem 4.8 by using Theorem 2.8. □

Example 4.9 We consider the following problem

$$\begin{cases}
D_{1^-}^{\frac{1}{4}}(D_{0^+}^{\frac{1}{4}}(D_{1^-}^{\frac{1}{4}}(D_{0^+}^{\frac{1}{4}}u(t))) = f(t,u(t))), & t \in [0,1], \\
u(0) = u(1) = 0, \\
D_{1^-}^{\frac{1}{4}}(D_{0^+}^{\frac{1}{4}}u(0)) = D_{1^-}^{\frac{1}{4}}(D_{0^+}^{\frac{1}{4}}u(T)) = 0,
\end{cases} \tag{4.4}$$

with

$$f(t,y) = \begin{cases} ty^{\frac{1}{2}}, & \text{if } y \geq 0 \\ t(-y)^{\frac{1}{2}}, & \text{if } y \leq 0 \end{cases}$$

$\alpha = \frac{1}{4}$ and

$$F(t,x) = \int_0^1 f(t,y)\,dy = \frac{2}{3}t|x|^{\frac{3}{2}}.$$

The hypotheses of Theorem 4.8 hold with $b(t) = \frac{1}{2} \in C([0,1],\mathbb{R})$, where $b^* = \frac{1}{2} > 0$. In fact

$$\limsup_{|x|\to+\infty}\frac{F(t,x)}{|x|^2} = 0 < \frac{1}{2}$$

with

$$\frac{1}{2} - b^* T\frac{T^{2\alpha-1}}{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)} = \frac{1}{2} + \frac{1}{\Gamma^2(\frac{1}{4})\Gamma^2(\frac{5}{4})} > 0.$$

Thus problem (4.4) has at least one solution.

Theorem 4.10 *Suppose that*

(H_1) *there exist* $\mu \in [0, \frac{1}{2})$ *and* $R > 0$ *such that* $0 < F(t,x) \leq \mu x f(t,x)$ *for all* $x \in \mathbb{R}$ *with* $|x| \geq R$ *and* $t \in [0,T]$ *and*

(H_2)
$$\limsup_{|x| \to 0} \frac{F(t,x)}{|x|^2} < \frac{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)}{2T^{4\alpha}}$$

uniformly with respect to $t \in [0,T]$ *hold,*

then the boundary value problem (1.1) *has at least one solution.*

Proof We will verify that φ satisfies all the conditions of Theorem (2.9). It is obvious by the definition of φ that $\varphi(0) = 0$.

Step 1: φ satisfies the condition of Palais–Smale.

Since $F(t,x) - \mu x f(t,x)$ is continuous for $t \in [0,T]$ and $|x| \leq R$, there exists $C \in \mathbb{R}^+$, such that

$$F(t,x) \leq \mu x f(t,x) + C, \quad \text{for } t \in [0,T] \text{ and } |x| \leq R.$$

By condition (H_3), we obtain that

$$F(t,x) \leq \mu x f(t,x) + C, \quad \text{for } t \in [0,T] \text{ and } x \in \mathbb{R}.$$

Let (u_n) be a sequence in $E_0^{2\alpha}$ such that $\lim_{n\to+\infty} \varphi'(u_n) = 0$ and $\varphi(u_n)$ is bounded, i.e.; $|\varphi(u_n)| \leq K$, for n so large. We note that

$$\langle \varphi'(u_n), u_n \rangle = \int_0^T \left[|D_{T^-}^{\alpha}(D_{0^+}^{\alpha} u_n(t))|^2 - f(t, u_n(t)).u_n(t) \right] dt. \tag{4.5}$$

In view of condition (H_1) and the relation (3.6), we have

$$\begin{aligned}
K \geq \varphi(u_n) &= \frac{1}{2} \int_0^T |D_{T^-}^{\alpha}(D_{0^+}^{\alpha} u_n(t))|^2 - \int_0^T F(t, u_n(t))\, dt \\
&= \frac{1}{2} \|u_n\|_{2\alpha}^2 - \int_0^T F(t, u_n(t))\, dt \\
&= \frac{1}{2} \|u_n\|_{2\alpha}^2 - \mu \int_0^T f(t, u_n(t)) u_n(t)\, dt - CT \\
&= \frac{1}{2} \|u_n\|_{2\alpha}^2 - \mu \big[\|u_n\|_{2\alpha}^2 - \langle \varphi'(u_n), u_n \rangle \big] - CT \\
&\geq (\frac{1}{2} - \mu) \|u_n\|_{2\alpha}^2 - \mu \|\varphi'(u_n)\|_{(E_0^{2\alpha})'} \|u_n\|_{2\alpha} - CT.
\end{aligned}$$

Since $\lim_{n\to+\infty} \varphi'(u_n) = 0$, there exists $n_0 \in \mathbb{N}$ such that $\|\varphi'(u_n)\|_{(E_0^{2\alpha})'} \leq 1$ for $n \geq n_0$. Then $K \geq (\frac{1}{2} - \mu)\|u_n\|_{2\alpha}^2 - \|u_n\|_{2\alpha} - CT$, $n \geq n_0$, which implies that (u_n) is bounded in $E_0^{2\alpha}$.

In fact, if (u_n) is unbounded, there exists $(u_{n_k}) \subset (u_n)$ such that

$$\lim_{k\to+\infty} \|u_{n_k}\|_{2\alpha} = +\infty;$$

then we obtain

$$K \geq \lim_{k\to+\infty} \Big[(\frac{1}{2} - \mu)\|u_{n_k}\|_{2\alpha}^2 - \|u_{n_k}\|_{2\alpha} - CT \Big] = +\infty$$

which is a contradiction.

Since $E_0^{2\alpha}$ is a reflexive space, passing to a subsequence, we can assume that $u_n \rightharpoonup u$ in $E_0^{2\alpha}$ and (u_n) is bounded in $E_0^{2\alpha}$, thus we have

$$\begin{aligned} \langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle &= \langle \varphi'(u_n), u_n - u\rangle - \langle \varphi'(u), u_n - u\rangle \\ &\leq \|\varphi'(u_n)\|_{(E^{2\alpha})'} \|u_n - u\|_{2\alpha} - \langle \varphi'(u), u_n - u\rangle \to 0 \qquad (4.6) \end{aligned}$$

as $n \to +\infty$. Moreover, by Proposition 3.17, $\lim_{n\to+\infty} \|u_n - u\|_\infty = 0$ and $\|u_n\|_\infty \leq M$, $\forall n$. We have also $u_n \to u$ in $C([0,T], \mathbb{R})$ implies that

$$f(t, u_n(t)) \longrightarrow f(t, u(t)), \quad \forall\, t \in [0,T]$$

and

$$|f(t, u_n(t))| \leq \sup_{y\in[-M,M]} |f(t,y)| = g(t) \in L^1([0,T]),$$

then by the Lebesgue's dominated convergence theorem, we have

$$\int_0^T f(t, u_n(t))\, dt \to \int_0^T f(t, u(t))\, dt \quad \text{as } n \to \infty \qquad (4.7)$$

Noting that for n so large, we have

$$\begin{aligned} 0 &\geq \langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle \\ &= \int_0^T D_{T^-}^\alpha (D_{0^+}^\alpha (u_n(t) - u(t))).D_{T^-}^\alpha (D_{0^+}^\alpha (u_n(t) - u(t)))\, dt \\ &\quad - \int_0^T \big(f(t, u_n(t)) - f(t, u(t))\big)\big(u_n(t) - u(t)\big)\, dt \\ &\geq -\Big| \int_0^T \big(f(t, u_n(t)) - f(t, u(t))\big)\, dt \Big| \|u_n - u\|_\infty + \|u_n - u\|_{2\alpha}^2. \end{aligned}$$

Combining (4.6) and (4.7), it is easy that $\|u_n - u\|_{2\alpha}^2 \to 0$ as $n \to +\infty$, and hence that $u_n \to u$ in $E_0^{2\alpha}$. Thus, the compactness condition is satisfied.

Step 2: φ satisfies the first geometric condition.

In fact, by condition (H_2) there exist $r > 0$ and $\varepsilon > 0$ such that

$$F(t,x) \leq \Big(\frac{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)}{2T^{4\alpha}} - \varepsilon \Big)|x|^2 \quad \text{for } |x| \leq r \text{ and } t \in [0,T].$$

We put

$$\rho = \frac{\Gamma(\alpha)\Gamma(\alpha+1)(2(\alpha-1)+1)^{\frac{1}{2}}}{T^{2\alpha-\frac{1}{2}}} r, \quad \sigma = \varepsilon \frac{T^{4\alpha}}{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)} \rho^2.$$

Then from (3.6) we obtain

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}} \cdot \frac{T^\alpha}{\Gamma(\alpha+1)} \|u\|_{2\alpha}$$

for all $u \in E_0^{2\alpha}$ with $\|u\|_{2\alpha} = \rho$. Therefore, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_{2\alpha}^2 - \int_0^T F(t, u(t))\, dt \\
&\geq \frac{1}{2}\|u\|_{2\alpha}^2 - \Big(\frac{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)}{2T^{4\alpha}} - \varepsilon\Big) \int_0^T |u(t)|^2\, dt \\
&\geq \frac{1}{2}\|u\|_{2\alpha}^2 - \Big(\frac{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)}{2T^{4\alpha}} - \varepsilon\Big) T \|u\|_\infty^2 \\
&= \frac{1}{2}\varepsilon\rho^2 - \Big(\frac{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)}{2T^{4\alpha}} - \varepsilon\Big) T \\
&\quad \times \Big(\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}} \cdot \frac{T}{\Gamma(\alpha+1)}\Big)^2 \rho^2 \\
&= \varepsilon \frac{T^{4\alpha}}{\Gamma^2(\alpha)\Gamma^2(\alpha+1)(2(\alpha-1)+1)} \rho^2 = \sigma.
\end{aligned}$$

for all $u \in E_0^{2\alpha}$ with $\|u\|_{2\alpha} = \rho$. This implies that (ii) in Theorem 2.9 is satisfied.

Step 3: φ satisfies the second geometric condition.

By condition (H_1), there exist $C_3, C_4 > 0$ such that

$$F(t,x) \geq C_3|x|^{\frac{1}{\mu}} + C_4 \quad \text{for all } t \in [0,T] \text{ and } \ x \in \mathbb{R}.$$

We have for all $u \in E_0^{2\alpha}$ with $u \neq 0$, $\eta > 0$ and noting that $\mu \in [0, \frac{1}{2})$, we obtain

$$\begin{aligned}
\varphi(\eta u) = \frac{1}{2}\|\eta u\|_{2\alpha}^2 - \int_0^T F(t, \eta u(t))\, dt &\leq \frac{\eta^2}{2}\|u\|_{2\alpha}^2 - C_3 \int_0^T |\eta u(t)|^{\frac{1}{\mu}} dt + C_4 T \\
&= \frac{\eta^2}{2}\|u\|_{2\alpha}^2 - C_3 \eta^{1/\mu} \|u\|_{L^{\frac{1}{\mu}}}^{\frac{1}{\mu}} + C_4 T \longrightarrow -\infty
\end{aligned}$$

as $\eta \to +\infty$. Then there exists η_0 large enough, such that $\varphi(\eta_0 u) < 0$. Consequently, the condition (iii) in Theorem 2.9 is satisfied. While for our critical point u, $\varphi(u) \geq \sigma > 0$. Hence u is a nontrivial solution of the boundary value problem (1.1). □

Example 4.11 We consider the following problem

$$\begin{cases} D^{\frac{4}{5}}_{1^-}(D^{\frac{4}{5}}_{0^+}(D^{\frac{4}{5}}_{1^-}(D^{\frac{4}{5}}_{0^+}u(t)))) = 3|u(t)|u(t), \quad t \in [0,T], \\ u(0) = u(1) = 0, \\ D^{\frac{4}{5}}_{1^-}(D^{\frac{4}{5}}_{0^+}u(0)) = D^{\frac{4}{5}}_{1^-}(D^{\frac{4}{5}}_{0^+}u(T)) = 0, \end{cases} \tag{4.8}$$

where $\alpha = \frac{4}{5}$, $f(t,y) = 3|y|y$ and

$$F(t,x) = \int_0^x f(t,y)\,dy = |y|^3.$$

We have (H_1) holds with $\mu = \frac{1}{3} \in [0, \frac{1}{2}[$. We put $T = 1$ and by the condition (H_2) we obtain

$$\limsup_{|x|\to 0} \frac{F(t,x)}{|x|^2} = \limsup_{|x|\to 0} \frac{|x|^3}{|x|^2} = 0$$
$$< \frac{\Gamma^2(\frac{4}{5})\Gamma^2(\frac{9}{5})(2(\frac{4}{5}-1)+1)}{2} = \Gamma^2\Big(\frac{4}{5}\Big)\Gamma^2\Big(\frac{9}{5}\Big)\Big(\frac{3}{10}\Big).$$

Thus problem (4.8) has at least one solution.

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