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Stability and Boundedness of the Solutions of Non Autonomous Third Order Differential Equations with Delay

Moussadek REMILI [a*], Lynda Damerdji OUDJEDI [b]

Department of Mathematics, University of Oran
31000 Oran, Algeria
[a] *e-mail: remilimous@gmail.com*
[b] *e-mail: oudjedi@yahoo.fr*

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Abstract

In this article, we shall establish sufficient conditions for the asymptotic stability and boundedness of solutions of a certain third order nonlinear non-autonomous delay differential equation, by using a Lyapunov function as basic tool. In doing so we extend some existing results. Examples are given to illustrate our results.

Key words: Stability, Lyapunov functional, delay differential equations, third-order differential equations.

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1 Introduction

In this article, we establish the uniform asymptotic stability of the equation of the form

$$[g(x(t))x'(t)]'' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = 0, \tag{1.1}$$

and the boundedness of

$$[g(x(t))x'(t)]'' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = p(t), \tag{1.2}$$

[*] Corresponding author

where $a(t), b(t), c(t), g(x), p(t)$, and $f(x)$ continuous functions depending only on the arguments shown and $g'(x)$, $f'(x)$ exist and are continuous for all x, $f(0) = 0$.

The author in [4, 5] based on the results in [8] have applied the method used in [8] to construct some new Lyapunov functions to examine the asymptotic stability and boundedness of the solutions of non-linear delay differential equation described by

$$x''' + a(t)x'' + b(t)x' + c(t)f(x(t-r)) = p(t), \tag{1.3}$$

with $p \equiv 0$ and $p \neq 0$, respectively.

The asymptotic stability and boundedness of solutions of this equation have been studied by a variety of authors over the years, and we mention only a sampling of such papers [1–15] and other references therein.

Obviously, the equation discussed in [4], Eq.(1), is a particular case of our equation (1.1). We shall use appropriate Lyapunov function and impose suitable conditions on the functions g and f.

2 Preliminaries

First, we will give the preliminary definitions and the stability criteria for the general non-autonomous delay differential system. We consider

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta), \ -r \leq \theta \leq 0, \ t \geq 0, \tag{2.1}$$

where $f\colon I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$, $C_H := \{\phi \in (C[-r, 0], \ \mathbb{R}^n)\colon \|\phi\| \leq H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t, \phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 2.1 [2] An element $\psi \in C$ is in the ω-limit set of ϕ, say $\Omega(\phi)$, if $x(t, 0, \phi)$ is defined on $[0, +\infty)$ and there is a sequence $\{t_n\}$, $t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \leq \theta \leq 0$.

Definition 2.2 [2] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t, 0, \phi)$, is defined on $[0, \infty)$ and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

Lemma 2.3 [1] *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2,1) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \quad as\ t \to \infty.$$

Lemma 2.4 [1] *Let $V(t, \phi)\colon I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t, 0) = 0$, and such that:*
(i) *$W_1(|\phi(0)|) \leq V(t, \phi) \leq W_2(\|\phi\|)$ where $W_1(r)$, $W_2(r)$ are wedges.*
(ii) *$V'_{(2,1)}(t, \phi) \leq 0$, for $\phi \in C_H$.*
Then the zero solution of (2.1) is uniformly stable.

If $Z = \{\phi \in C_H\colon V'_{(2,1)}(t, \phi) = 0\}$, then the zero solution of (2.1) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3 Assumptions and main results

We shall state here some assumptions which will be used on the functions that appeared in equation (1.1), and suppose that there are positive constants a_0, b_0, c_0, d, A, B, C, and ε, such that the following conditions are satisfied:

i) $0 < a_0 \le a(t) \le A$; $0 < b_0 \le b(t) \le B$; $0 < c_0 \le c(t) \le C$.

ii) $c(t) \le b(t)$, $-L \le b'(t) \le c'(t) \le 0$ for $t \in [0, \infty)$.

iii) $0 < m \le g(x) \le M$.

iv) $f(0) = 0$, $\frac{f(x)}{x} \ge \delta_0 > 0$ $(x \ne 0)$, and $|f'(x)| \le \delta_1$ for all x.

v) $M\delta_1 < d < a_0$.

vi) $\frac{1}{2} da'(t) - b_0(d - M\delta_1) \le -\varepsilon < 0$.

vii) $\int_{-\infty}^{+\infty} |g'(u)|\, du < \infty$.

To simplify the notation in what follows, we let

$$\theta(t) = \frac{g'(x(t))}{g^2(x(t))} x'(t).$$

Theorem 3.1 *Suppose that assumptions (i) through (vii) hold. Then the solution $x(t)$ of (1.1) and their derivatives $x'(t)$ and $x''(t)$ are uniformly asymptotically stable, provided that there exists r satisfying*

$$r < \min\left\{ \frac{2(a_0 - d)}{MC\delta_1}, \frac{2m^3\varepsilon}{C\delta_1 M^2(d + dm^2 + m)} \right\}.$$

Proof We note that equation (1.1) is equivalent to the following system of differential equation

$$\begin{aligned} x' &= \frac{1}{g(x)} y \\ y' &= z \\ z' &= -\frac{a(t)}{g(x)} z + \frac{a(t)g'(x)}{g^3(x)} y^2 - \frac{b(t)y}{g(x)} - c(t)f(x) + c(t)\int_{t-r}^{t} y(s)\frac{f'(x(s))}{g(x(s))} ds. \end{aligned} \tag{3.1}$$

We define the Lyapunov functional $U = U(t, x_t, y_t, z_t)$ as

$$U(t, x_t, y_t, z_t) = \exp\left(-\frac{\gamma(t)}{\mu}\right) V(t, x_t, y_t, z_t) = \exp(-\frac{\gamma(t)}{\mu})V, \tag{3.2}$$

where $\gamma(t) = \int_0^t |\theta(s)|\, ds$, and

$$\begin{aligned} V &= dc(t)F(x) + c(t)f(x)y + \frac{b(t)}{2g(x)} y^2 + \frac{1}{2} z^2 + \frac{d}{g(x)} yz \\ &\quad + \frac{1}{2}\frac{da(t)}{g^2(x)} y^2 + \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)\, d\xi ds, \end{aligned} \tag{3.3}$$

such that $F(x) = \int_0^x f(u)du$, μ and λ are positives constants which will be determined later. From the definition of V in (3.3), we observe that the above Lyapunov functional can be rewritten as follows

$$V = V_1 + V_2 + \lambda \int_{-r}^{0} \int_{t+s}^{t} y^2(\xi)\, d\xi ds,$$

with

$$V_1 = dc(t)F(x) + c(t)f(x)y + \frac{b(t)}{2g(x)} y^2,$$

and

$$V_2 = \frac{1}{2} z^2 + \frac{d}{g(x)} yz + \frac{da(t)}{2g^2(x)} y^2.$$

We shall write the above expression as

$$V_2 = \frac{1}{2}\left\{ z^2 + \frac{2d}{g(x)} yz + \frac{da(t)}{g^2(x)} y^2 \right\} = \frac{1}{2}\left(z + \frac{d}{g(x)} y \right)^2 + \frac{d(a(t)-d)}{2g^2(x)} y^2.$$

By (v),

$$\frac{d(a(t)-d)}{2g^2(x)} \geq \frac{d(a_0-d)}{2g^2(x)} > 0.$$

Thus there exist positive constants such that

$$V_2 \geq \delta_2 y^2 + \delta_3 z^2. \tag{3.4}$$

On the other hand, using the assumptions (i)–(v), and a rearranged V_1, we obtain,

$$\begin{aligned} V_1 &= dc(t)F(x) + \frac{b(t)}{2g(x)} \left\{ y + \frac{c(t)f(x)g(x)}{b(t)} \right\}^2 - \frac{c^2(t)g(x)f^2(x)}{2b(t)} \\ &\geq dc(t)F(x) - \frac{c^2(t)g(x)f^2(x)}{2b(t)} \\ &\geq dc(t)[F(x) - \frac{M}{2d} f^2(x)] \\ &\geq dc(t) \int_0^x (1 - \frac{M\delta_1}{d}) f(u)\, du \\ &\geq \delta_4 \int_0^x f(u)\, du, \end{aligned}$$

where

$$\delta_4 = dc_0(1 - \frac{M\delta_1}{d}) > dc_0(1 - \frac{d}{d}) = 0.$$

Thus from (iv) we obtain,

$$V_1 \geq \frac{\delta_4 \delta_0}{2} x^2. \tag{3.5}$$

Clearly, from (3.5), (3.4) and (3.3), we have

$$V \geq \delta_2 y^2 + \delta_3 z^2 + \frac{\delta_4\delta_0}{2}x^2 + \lambda \int_{-r}^{0}\int_{t+s}^{t} y^2(\xi)\,d\xi ds.$$

Hence, it is evident, from the terms contained in the last inequality, that there exist sufficiently small positive constant k, such that

$$V \geq k(x^2 + y^2 + z^2), \tag{3.6}$$

since the integral $\int_{t+s}^{t} y^2(\xi)\,d\xi$ is positive, where $k = \min\{\delta_2; \delta_3; \frac{\delta_4\delta_0}{2}\}$.

Observe that by (iii) and (vii), we get

$$\gamma(t) = \int_0^t |\theta(s)|\,ds = \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|g'(u)|}{g^2(u)}\,du \leq \frac{1}{m^2}\int_{-\infty}^{+\infty} |g'(u)|\,du \leq N < \infty,$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$, and $\alpha_2(t) = \max\{x(0), x(t)\}$.

Therefore we can find a continuous function $W_1(|\Phi(0)|)$ with

$$W_1(|\Phi(0)|) \geq 0 \quad \text{and} \quad W_1(|\Phi(0)|) \leq U(t, \Phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $U(t, \phi) \leq W_2(\|\phi\|)$, is easily verified.

For the time derivative of the Lyapunov functional V, along the trajectories of the system (3.1), we have

$$\begin{aligned}
\frac{d}{dt}V \;&= dc'(t)F(x) + c'(t)yf(x) + \frac{b'(t)}{2g(x)}y^2 + \frac{1}{g(x)}(d - a(t))z^2 \\
&\quad + \frac{g'(x)x'}{g^2(x)}\left[(a(t) - d)zy - \frac{b(t)}{2}y^2\right] \\
&\quad + \left[\frac{da'(t) + 2c(t)g(x)f'(x) - 2db(t)}{2g^2(x)}\right]y^2 + \lambda r y^2 \\
&\quad + c(t)(z + \frac{d}{g(x)}y)\int_{t-r}^{t} y(s)\frac{f'(x(s))}{g(x(s))}\,ds - \lambda\int_{t-r}^{t} y^2(\xi)\,d\xi.
\end{aligned}$$

Consequently by the hypothesis (i)–(vi), we get

$$\begin{aligned}
\frac{d}{dt}V \;&\leq dc'(t)F(x) + c'(t)yf(x) + \frac{b'(t)}{2g(x)}y^2 \\
&\quad + |\theta(t)|\left[(A - d)\,|zy| + \frac{B}{2}y^2\right] - \left(\frac{\varepsilon}{M^2} - \lambda r\right)y^2 - \frac{1}{M}(a_0 - d)z^2 \\
&\quad + c(t)(z + \frac{dy}{g(x)})\int_{t-r}^{t} y(s)\frac{f'(x(s))}{g(x(s))}\,ds - \lambda\int_{t-r}^{t} y^2(\xi)\,d\xi.
\end{aligned}$$

We define the function H as

$$H(t, x, y) = dc'(t)F(x) + c'(t)yf(x) + \frac{b'(t)}{2g(x)}y^2,$$

for all x, y and $t \geq 0$. If $c'(t) = 0$, then

$$H(t,x,y) = \frac{b'(t)}{2g(x)}y^2 \leq 0.$$

If $c'(t) < 0$, the quantity $H(t,x,y)$ can be written as,

$$H(t,x,y) = dc'(t)H_1(t,x,y),$$

where

$$H_1(t,x,y) \equiv \left[F(x) + \frac{b'(t)}{2dg(x)c'(t)}\left\{y + \frac{c'(t)g(x)}{b'(t)}f(x)\right\}^2 - \frac{c'(t)g(x)}{2db'(t)}f^2(x)\right],$$

by assumption (ii) we have $0 < \frac{c'(t)}{b'(t)} \leq 1$, this implies

$$\begin{aligned} H_1(t,x,y) &\geq F(x) - \frac{g(x)}{2d}f^2(x) \geq F(x) - \frac{M}{2d}f^2(x) \\ &\geq \int_0^x (1 - \frac{M\delta_1}{d})f(u)\,du \geq \frac{\delta_4}{dc_0}\int_0^x f(u)\,du \geq 0. \end{aligned}$$

It follows immediately that

$$H(t,x,y) = dc'(t)H_1(t,x,y) \leq 0.$$

Hence, on combining the two cases, we have $H(t,x,y) \leq 0$ for all $t \geq 0$, x and y. Using the Schwartz inequality $|uv| \leq \frac{1}{2}(u^2 + v^2)$, we obtain

$$\begin{aligned} |\theta(t)|\left[(A-d)\,|zy| + \frac{B}{2}y^2\right] &\leq |\theta(t)|\left[\frac{A-d}{2}z^2 + \frac{A-d+B}{2}y^2\right] \\ &\leq k_1\,|\theta(t)|\,(y^2 + z^2), \end{aligned}$$

where $k_1 = \frac{A-d+B}{2}$. Since $|f'(x)| \leq \delta_1$, we obtain the following inequalities

$$\frac{dc(t)}{g(x)}y\int_{t-r}^t \frac{y(s)}{g(x(s))}f'(x(s))\,ds \leq \frac{C\delta_1 dr}{2m}y^2 + \frac{Cd\delta_1}{2m^3}\int_{t-r}^t y^2(\xi)\,d\xi,$$

and

$$c(t)z\int_{t-r}^t \frac{y(s)}{g(x(s))}f'(x(s))ds \leq \frac{C\delta_1 r}{2}z^2 + \frac{C\delta_1}{2m^2}\int_{t-r}^t y^2(\xi)d\xi.$$

With some rearrangements, we get

$$\begin{aligned} \frac{d}{dt}V \leq &-\left[\frac{\varepsilon}{M^2} - (\lambda + \frac{dC\delta_1}{2m})r\right]y^2 - \left[\frac{a_0 - d}{M} - \frac{C\delta_1 r}{2}\right]z^2 \\ &+ k_1\,|\theta(t)|\,(y^2 + z^2) + \left[\frac{C\delta_1}{2m^2}(1 + \frac{d}{m}) - \lambda\right]\int_{t-r}^t y^2(\xi)\,d\xi. \end{aligned}$$

If we take $\frac{C\delta_1}{2m^2}(1+\frac{d}{m})=\lambda$, the last inequality becomes

$$\frac{d}{dt}V \le -\left[\frac{\varepsilon}{M^2}-\frac{C\delta_1}{2m}(d+\frac{1}{m}+\frac{d}{m^2})r\right]y^2-\left[\frac{a_0-d}{M}-\frac{C\delta_1 r}{2}\right]z^2 + k_1\,|\theta(t)|\,(y^2+z^2).$$

Using (3.2), (3.6) and taking $\mu=\frac{k}{k_1}$ yields

$$\begin{aligned}\frac{d}{dt}U &= \exp\left(-\frac{k_1\gamma(t)}{k}\right)\left(\frac{d}{dt}V-\frac{k_1|\theta(t)|}{k}V\right)\\ &\le \exp\left(-\frac{k_1\gamma(t)}{k}\right)\left[-\left(\frac{\varepsilon}{M^2}-\frac{C\delta_1}{2m}\left(d+\frac{1}{m}+\frac{d}{m^2}\right)r\right)y^2\right.\\ &\quad\left.-\left(\frac{a_0-d}{M}-\frac{C\delta_1 r}{2}\right)z^2\right]. \end{aligned}\tag{3.7}$$

Therefore, if

$$r<\min\left\{\frac{2(a_0-d)}{MC\delta_1},\frac{2m^3\varepsilon}{C\delta_1 M^2(d+dm^2+m)}\right\},$$

the inequality (3.7) becomes

$$\frac{d}{dt}U(t,x_t,y_t,z_t)\le-\beta\exp\left(-\frac{k_1N}{k}\right)(y^2+z^2),\quad \text{for some } \beta>0.$$

It is clear that the largest invariant set in Z is $Q=\{0\}$, where

$$Z=\left\{\phi\in C_H\colon \frac{d}{dt}U(\phi)=0\right\}.$$

Namely, the only solution of system (3.1) for which $\frac{d}{dt}U(t,x_t,y_t,z_t)=0$ is the solution $x=y=z=0$. Thus, under the above discussion, we conclude that the trivial solution of equation (1.1) is uniformly asymptotically stable. This fact completes the proof. □

4 Example

In this section, we give example to illustrate our main results.
We consider the following third order non-autonomous delay differential equation

$$\begin{aligned}&\left[\left(\frac{\sin x}{1+x^2}+2\right)x'\right]''+\left(\frac{1}{4}\sin t+\frac{1}{2}\right)x''+\left(\frac{1}{2+t^2}+1\right)x'\\ &\quad+\frac{1}{28}\left(\frac{1}{3+t^2}+\frac{1}{4}\right)\left(x(t-r)+\frac{x(t-r)}{1+x^2(t-r)}\right)=0.\end{aligned}\tag{4.1}$$

Now, it is easy to see that

$$\frac{1}{4} = a_0 \leq a(t) = \frac{1}{4}\sin t + \frac{1}{2} \leq \frac{3}{4},$$

$$a'(t) = \frac{1}{4}\cos t \leq \frac{1}{4} \text{ for all } t \geq 0,$$

$$1 = b_0 \leq b(t) = \frac{1}{2+t^2} + 1 \leq \frac{3}{2},$$

$$\frac{1}{4} \leq c(t) = \frac{1}{3+t^2} + \frac{1}{4} \leq \frac{7}{12} = C,$$

$$1 \leq g(x) = \frac{\sin x}{1+x^2} + 2 \leq 3 = M,$$

$$\frac{1}{28} \leq \frac{f(x)}{x} = \frac{1}{28}\left(1 + \frac{1}{1+x^2}\right) \text{ with } x \neq 0, \text{ and } |f'(x)| \leq \frac{1}{14} = \delta_1,$$

$$M\delta_1 = \frac{3}{14} < d < \frac{1}{4} = a_0,$$

$$\frac{1}{2}a'(t) = \frac{1}{8}\cos t < b_0\left(1 - \frac{M\delta_1}{d}\right) < \frac{1}{7}.$$

A sample calculation shows

$$\int_{-\infty}^{+\infty} |g'(u)|\, du \leq \int_{-\infty}^{+\infty} \left[\left|\frac{\cos u}{1+u^2}\right| + \left|\frac{2u\sin u}{(1+u^2)^2}\right|\right] du \leq \pi + 2.$$

All the assumptions (i) through (vii) are satisfied, we can conclude using Theorem 3.1 that every solution of (4.1) is uniformly asymptotically stable.

In the case $p(t) \neq 0$ we establish the following result:

Theorem 4.1 *In addition to the assumptions of Theorem* 3.1*, If we assume that* $p(t)$ *is continuous in* $\mathbb{R}$ *and*

$$\int_0^t p(s)ds < \infty \quad \textit{for all } t \geq 0,$$

then all solutions of the perturbed equation (1.2) *are bounded.*

Proof The proof of this theorem is similar to that of the proof of Theorem 2 in [5] and hence it is omitted. □

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References

[1] Burton, T. A.: Stability and periodic solutions of ordinary and functional differential equations. Mathematics in Science and Engineering **178**, *Academic Press*, Orlando, FL, 1985.

[2] Burton, T. A., Makay, G.: *Asymptotic stability for functional differential equations.* Acta Math. Hung. **65**, 3 (1994), 243–251.

[3] Greaf, J. R., Remili, M.: *Some properties of monotonic solutions of* $x''' + p(t)x' + q(t)f(x) = 0$. Pan. American Math. J. **22**, 2 (2012), 31–39.

[4] Omeike, M. O.: *New results on the stability of solution of some non-autonomous delay differential equations of the third order*. Differential Equations and Control Processes **2010**, 1 (2010), 18–29.

[5] Omeike, M. O.: *Stability and boundedness of solutions of some non-autonomous delay differential equation of the third order*. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **55**, 1 (2009), 49–58.

[6] Sadek, A. I.: *On the Stability of Solutions of Some Non-Autonomous Delay Differential Equations of the Third Order*. Asymptot. Anal. **43**, 1-2 (2005), 1–7.

[7] Sadek, A. I.: *Stability and Boundedness of a Kind of Third-Order Delay Differential System*. Applied Mathematics Letters **16**, 5 (2003), 657–662.

[8] Swick, K.: *On the boundedness and the stability of solutions of some non- autonomous differential equations of the third order*. J. London Math. Soc. **44** (1969), 347–359.

[9] Tunç, C.: *On asymptotic stability of solutions to third order nonlinear differential equations with retarded argument*. Communications in applied analysis **11**, 4 (2007), 515–528.

[10] Tunç, C.: *On the asymptotic behavior of solutions of certain third-order nonlinear differential equations*. J. Appl. Math. Stoch. Anal. **2005**, 1 (2005), 29–35.

[11] Tunç, C.: *On the stability and boundedness of solutions to third order nonlinear differential equations with retarded argument*. Nonlinear Dynamics **57** (2009), 97–106.

[12] Tunç, C.: *Some stability and boundedness conditions for non-autonomous differential equations with deviating arguments*. E. J. Qualitative Theory of Diff. Equ. **2010**, 1 (2005), 1–12.

[13] Tunç, C.: *Stability and boundedness of solutions of nonlinear differential equations of third-order with delay*. Differential Equations and Control Processes **2007**, 3 (2007), 1–13.

[14] Yoshizawa, T.: Stability theory by Liapunov's second method. *The Mathematical Society of Japan*, Tokyo, 1966.

[15] Zhu, Y. F.: *On stability, boundedness and existence of periodic solution of a kind of third order nonlinear delay differential system*. Ann. Differential Equations **8**, 2 (1992), 249–259.