Orthomodular Posets Can Be Organized as Conditionally Residuated Structures*

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Abstract

It is proved that orthomodular posets are in a natural one-to-one correspondence with certain residuated structures.

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Orthomodular posets are well-known structures used in the foundations of quantum mechanics (cf. e.g. [4], [5], [9], [10] and [11]). They can be considered as effect algebras (see e.g. [6]). Residuated lattices were treated in [7]. In [3] the concept of a conditionally residuated structure was introduced. Since every orthomodular poset is in fact an effect algebra, it follows that also every orthomodular poset can be considered as a conditionally residuated structure. The question is which additional conditions have to be satisfied in order to get a one-to-one correspondence. Contrary to the case of effect algebras, orthomodular posets satisfy also the orthomodular law and a certain condition concerning the orthogonality of their elements.

We start with the definition of an orthomodular poset.

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Definition 1 An orthomodular poset (cf. [8], [2] and [12]) is an ordered quintuple \( P = (P, \leq, ^+, 0, 1) \) where \( (P, \leq, 0, 1) \) is a bounded poset, \( ^+ \) is a unary operation on \( P \) and the following conditions hold for all \( x, y \in P \):

(i) \( (x^+)^+ = x \)
(ii) If \( x \leq y \) then \( y^+ \leq x^+ \).
(iii) If \( x \perp y \) then \( x \lor y \) exists.
(iv) If \( x \leq y \) then \( y = x \lor (y \land x^+) \).

Here and in the following \( x \perp y \) is an abbreviation for \( x \leq y^+ \).

Remark 2 If \( (P, \leq) \) is a poset and \( ^+ \) a unary operation on \( P \) satisfying (i) and (ii) then the so-called de Morgan laws

\[
(x \lor y)^+ = x^+ \land y^+ \text{ in case } x \perp y \text{ and }
\]
\[
(x \land y)^+ = x^+ \lor y^+ \text{ in case } x^+ \perp y^+ 
\]

hold. Moreover, (iv) is equivalent to the following condition:

(v) If \( x \leq y \) then \( x = y \land (x \lor y^+) \).

If \( x \leq y \) then \( x \perp y^+ \) and therefore \( x \lor y^+ \) is defined. Hence also \( y \land x^+ \) is defined. Moreover, \( x \perp y \land x^+ \) which shows that \( x \lor (y \land x^+) \) is defined. Thus the expression in (iv) is well-defined. The same is true for condition (v).

Next we define a partial commutative groupoid with unit.

Definition 3 A partial commutative groupoid with unit is a partial algebra \( A = (A, \odot, 1) \) of type \((2, 0)\) satisfying the following conditions for all \( x, y \in A \):

(i) \( x \odot y \) is defined so is \( y \odot x \) and \( x \odot y = y \odot x \).
(ii) \( x \odot 1 \) and \( 1 \odot x \) are defined and \( x \odot 1 = 1 \odot x = x \).

Now we are ready to define a conditionally residuated structure.

Definition 4 Let \( A = (A, \leq, \odot, \to, 0, 1) \) be an ordered sixtuple such that \( (A, \leq, 0, 1) \) is a bounded poset, \( (A, \odot, \to, 0, 1) \) is a partial algebra of type \((2, 2, 0, 0)\), \( (A, \odot, 1) \) is a partial commutative groupoid with unit and \( x \to y \) is defined if and only if \( y \leq x \). We write \( x' \) instead of \( x \to 0 \). Moreover, assume that the following conditions are satisfied for all \( x, y, z \in A \):

(i) \( x \odot y \) is defined if and only if \( x' \leq y \).
(ii) If \( x \odot y \) and \( y \to z \) are defined then \( x \odot y \leq z \) if and only if \( x \leq y \to z \).
(iii) If \( x \to y \) is defined then so is \( y' \to x' \) and \( x \to y = y' \to x' \).
(iv) If \( y \leq x \) and \( x', y \leq z \) then \( x \to y \leq z \).

Then \( A \) is called a conditionally residuated structure.
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Remark 5 Condition (ii) is called left adjointness, see e.g. [1].

Example 6 Let $M := \{1, \ldots, 6\}$ and $P := \{C \subseteq M \mid |C| \text{ is even}\}$. If one defines for arbitrary $A, B \in P$

$$A \odot M = M \odot A := A,$$
$$A \odot (M \setminus A) := \emptyset,$$
$$A \odot B := A \cap B \text{ if } |A| = |B| = 4 \text{ and } A \cup B = M,$$
$$A \rightarrow \emptyset := M \setminus A,$$
$$A \rightarrow A := M,$$
$$M \rightarrow A := A$$

and

$$A \rightarrow B := (M \setminus A) \cup B \text{ if } B \subseteq A, |B| = 2 \text{ and } |A| = 4$$

then $(P, \subseteq, \odot, \rightarrow, \emptyset, M)$ is a conditionally residuated structure.

The following lemma lists some easy properties of conditionally residuated structures used later on.

Lemma 7 If $\mathcal{A} = (A, \leq, \odot, \rightarrow, 0, 1)$ is a conditionally residuated structure then the following conditions hold for all $x, y \in A$:

(i) $(x')' = x$

(ii) If $x \leq y$ then $y' \leq x'$.

(iii) If $x \odot y$ is defined then $x \odot y = 0$ if and only if $x \leq y'$.

(iv) $x \rightarrow y = 1$ if and only if $x \leq y$.

Proof Let $x, y \in A$. We have $x' \leq x'$. Hence $x \odot x'$ exists and therefore also $x' \odot x$ exists which implies $(x')' \leq x$. Moreover, $x' \leq x' = x \rightarrow 0$ and hence $x' \odot x \leq 0$ which shows $x' \odot x = 0$ whence $x \odot x' = 0$. Now $x \odot x' \leq 0$ implies $x \leq x' \rightarrow 0 = (x')'$. Together we obtain $(x')' = x$. The inequality $x \leq y$ implies that $x' \odot y$ exists. Hence $y \odot x'$ exists wherefrom we conclude that $y' \leq x'$. Moreover, if $x \odot y$ is defined then the following are equivalent: $x \odot y = 0, x \odot y \leq 0, x \leq y \rightarrow 0, x \leq y'$. Finally, the following are equivalent: $x \rightarrow y = 1, 1 \leq x \rightarrow y, 1 \odot x \leq y, x \leq y$.

We now introduce two more properties of conditionally residuated structures.

Definition 8 A conditionally residuated structure $\mathcal{A} = (A, \leq, \odot, \rightarrow, 0, 1)$ is said to satisfy the divisibility condition if $y \leq x$ implies that $x \odot (x \rightarrow y)$ exists and $x \odot (x \rightarrow y) = y$ and it is said to satisfy the orthogonality condition if $x \leq y'$, $y \leq z'$ and $z \leq x'$ together imply $z \leq x' \odot y'$.

In the following theorem we show that an orthomodular poset can be considered as a special conditionally residuated structure.
Theorem 9 If $\mathcal{P} = (P, \leq, \perp, 0, 1)$ is an orthomodular poset and one defines

\[
x \circ y := x \land y \text{ if and only if } x \perp \leq y \text{ and } \ \ \ x \to y := x \perp \lor y \text{ if and only if } y \leq x
\]

for all $x, y \in P$ then $A(\mathcal{P}) := (P, \leq, \circ, \to, 0, 1)$ is a conditionally residuated structure satisfying both the divisibility and orthogonality condition.

Proof Let $a, b, c \in P$. Of course, $(P, \leq, 0, 1)$ is a bounded poset. The operations $\circ$ and $\to$ are well-defined since $a \perp \leq b$ implies $a \perp \perp b \perp$ and $b \leq a$ implies $a \perp \perp b$. If $a \circ b$ is defined then $a \perp \leq b$ and hence $b \perp \leq a$ which shows that $b \circ a$ is defined and $a \circ b = a \land b = b \land a = b \circ a$. Since $a \perp \leq 1$ we have that $a \perp 1$ is defined and $a \perp 1 = a \land 1 = a$. Because of $1 \perp = 0 \leq a$ we have that $1 \perp a$ is defined and $1 \perp a = 1 \land a = a$ showing that $(P, \circ, 1)$ is a partial commutative groupoid with unit. Now assume that $a \circ b$ and $b \to c$ are defined. Then $a \perp \leq b$ and $c \leq b$. If $a \circ b \leq c$ then $a \geq b \perp$ and

\[
a = b \perp \lor (a \land b) = b \perp \lor (a \circ b) \leq b \perp \lor c = b \to c.
\]

If, conversely, $a \leq b \to c$ then $c \leq b$ and

\[
a \circ b = a \land b \leq (b \to c) \land b = (b \perp \lor c) \land b = c.
\]

This proves left adjointness. If $b \leq a$ then $a \perp \leq b \perp$ and

\[
a \to b = a \perp \lor b = b \lor a \perp = b \perp \to a \perp.
\]

If $b \leq a$ and $a \perp, b \leq c$ then $a \to b = a \perp \lor b \leq c$. If $b \leq a$ then $a \to b$ exists and $a \perp \leq a \perp \lor b = a \to b$ and hence $a \circ (a \to b)$ exists and, by (v) of Remark 2, $a \circ (a \to b) = a \land (a \perp \lor b) = b$ showing that $A(\mathcal{P})$ satisfies the divisibility condition. Finally, if $a \leq b \perp$, $b \leq c \perp$ and $c \leq a \perp$ then there exists $a \perp \circ b \perp = a \perp \land b \perp$, $c \leq a \perp$ and $c \leq b \perp$ and hence $c \leq a \perp \land b \perp = a \perp \circ b \perp$ showing that $A(\mathcal{P})$ satisfies the orthogonality condition. \hfill \square

Conversely, we show that certain conditionally residuated structures can be converted in an orthomodular poset.

Theorem 10 If $\mathcal{A} = (A, \leq, \circ, \to, 0, 1)$ is a conditionally residuated structure satisfying the divisibility and orthogonality condition then $P(\mathcal{A}) := (A, \leq, ', 0, 1)$ is an orthomodular poset.

Proof Let $a, b, c \in A$. Of course, $(A, \leq, 0, 1)$ is a bounded poset. According to Lemma 7, the operation $'$ is an antitone involution of $(A, \leq)$. We show that in case $a \leq b'$ we have $(a' \circ b')' = a \lor b$. If $a \leq b'$ then $a' \circ b'$ and $b' \circ a'$ are defined. Now we have $b' \leq 1 = a' \to a'$ according to Lemma 7, hence $a' \circ b' = b' \circ a' \leq a'$ and therefore $a \leq (a' \circ b')'$. By symmetry $b \leq (a' \circ b')'$ follows. Now, if $a, b \leq c$ then $a \leq b'$, $b \leq c$ and $c \leq a'$ and hence according to the orthogonality condition $c' \leq a' \circ b'$ whence $c \geq (a' \circ b')'$. This shows $(a' \circ b')' = a \lor b$ in case $a \leq b'$. Since $a \leq (a')'$ we have $a \lor a' = (a' \circ a)' = 0' = 1$
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according to Lemma 7. Finally, assume $a \leq b$. Because of $a' \rightarrow b' \geq a' \rightarrow 0 = a$ and $a' \rightarrow b' \geq 1 \rightarrow b' = b'$ we have $a' \rightarrow b' \geq a \vee b'$. Hence, according to the divisibility condition we obtain

$$a \vee (b \wedge a') = (a' \circ (a \vee b'))' \geq (a' \circ (a' \rightarrow b'))' = (b')' = b.$$  

Since the converse inequality is obvious, we see that the considered poset is orthomodular. \hfill \square

Finally, we show that the correspondence described in the last two theorems is one-to-one.

**Theorem 11** If $\mathcal{P} = (P, \leq, \perp, 0, 1)$ is an orthomodular poset then $\mathbf{P}(\mathbf{A}(\mathcal{P})) = \mathcal{P}$. If $\mathcal{A} = (A, \leq, \odot, \rightarrow, 0, 1)$ is a conditionally residuated structure satisfying the divisibility and orthogonality condition then $\mathbf{A}(\mathbf{P}(\mathcal{A})) = \mathcal{A}$.

**Proof** First assume $\mathcal{P} = (P, \leq, \perp, 0, 1)$ to be an orthomodular poset and let $\mathbf{A}(\mathbf{P}) = (P, \leq, \odot, \rightarrow, 0, 1)$ and $\mathbf{P}(\mathbf{A}(\mathcal{P})) = (P, \leq, \star, 0, 1)$. Then

$$x^\star = x \rightarrow 0 = x \perp \vee 0 = x \perp$$

for all $x \in P$ and hence $\mathbf{P}(\mathbf{A}(\mathcal{P})) = \mathcal{P}$.

Conversely, assume $\mathcal{A} = (A, \leq, \odot, \rightarrow, 0, 1)$ to be a conditionally residuated structure satisfying the divisibility and orthogonality condition and let $\mathbf{P}(\mathcal{A}) = (A, \leq, \prime, 0, 1)$ and $\mathbf{A}(\mathbf{P}(\mathcal{A})) = (A, \leq, \odot, \rightarrow, 0, 1)$. Let $a, b, c \in A$. If $a' \leq b$ then $a \odot b = a \wedge b = (a' \vee b')' = a \circ b$ according to the proof of Theorem 10. Finally, if $b \leq a$ then $a \Rightarrow b = a' \vee b = a \rightarrow b$ according to the proof of Theorem 10. \hfill \square

**References**


