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Ulam Stabilities for Partial Impulsive Fractional Differential Equations

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Abstract

In this paper we investigate the existence of solutions for the initial value problems (IVP for short), for a class of implicit impulsive hyperbolic differential equations by using the lower and upper solutions method combined with Schauder's fixed point theorem.

Key words: fractional differential equations, impulse, Caputo fractional order derivative, left-sided mixed Riemann–Liouville integral, Darboux problem, Ulam stability

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1 Introduction

The fractional calculus deals with extensions of derivatives and integrals to non-integer orders. It represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering ([12, 18, 25, 28, 33]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [8], Baleanu *et al.* [12], Kilbas *et al.* [24], Lakshmikantham *et al.*, the papers of Abbas *et al.* [1, 2, 3, 4, 5, 6, 7, 9, 10], Ahmad and Nieto [11], Benchohra *et al.* [13], Cabada and Stanek [15], Kilbas and Marzan [23], Stanek [32], Vityuk and Golushkov [35], Wang *et al.* [37] and the references therein.

There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations with fixed moments. Recently some results on the Darboux problem for fractional order impulsive hyperbolic differential equations and inclusions have been obtained by Abbas *et al.* [4, 6, 8]. In [6], Abbas *et al.* studied the existence and the uniqueness of solutions of the following Darboux problem of partial impulsive differential equations

$$\begin{cases} {}^cD^r_{\theta_k}u(x,y) = f(x,y,u(x,y)); \ \text{ if } (x,y)\in J_k, \ k=0,\dots,m, \\ u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)); \ \text{ if } y\in[0,b], \ k=1,\dots,m, \\ u(x,0)=\varphi(x); \ x\in[0,a], \\ u(0,y)=\psi(y); \ y\in[0,b], \\ \varphi(0)=\psi(0), \end{cases} \tag{1}$$

where $J_0 = [0,x_1]\times[0,b]$, $J_k := (x_k,x_{k+1}]\times[0,b]$; $k=1,\dots,m$, $a,b>0$, $\theta_k=(x_k,0)$; $k=0,\dots,m$, ${}^cD^r_{\theta_k}$ is the fractional Caputo derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1]$, $0=x_0<x_1<\cdots<x_m<x_{m+1}=a$, $f\colon J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$, $J=[0,a]\times[0,b]$, $I_k\colon\mathbb{R}^n\to\mathbb{R}^n$; $k=1,\dots,m$ are given continuous functions, $\varphi\colon[0,a]\to\mathbb{R}^n$ and $\psi\colon[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions. Here $u(x_k^+,y)$ and $u(x_k^-,y)$ denote the right and left limits of $u(x,y)$ at $x=x_k$, respectively.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (for more details see [34]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [19]. Thereafter, this type of stability is called the Ulam–Hyers stability. In 1978, Rassias [29] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation?, or equivalently for every solution of the perturbed equation there exists a solution of the equation that is close to it. Considerable attention has been given to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; one can see the monographs of [20, 21]. Bota-Boriceanu and Petrusel [14], Petru *et al.* [26, 27], and Rus [30, 31] discussed the Ulam–Hyers stability for operatorial equations and inclusions. Castro and Ramos [16], and Jung [22] considered the Hyers–Ulam–Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [38, 39]. Some stability results for fractional integral equation are obtained by Wei *et al.* [42]. In [36, 41], Wang *et al.* introduced some new concepts about Ulam stability of impulsive differential equations with integer and non integer order. Mittag–Leffler–Ulam stabilities of fractional evolution equations have been considered by Wang and Zhou [40]. More details from historical point of view, and recent developments of such stabilities are reported in [21, 30, 42].

Motivated by the above papers, in this article, we discuss the Ulam stability for impulsive fractional partial differential equations

$$\begin{cases} {}^cD^r_{x_k}u(x,y) = f(x,y,u(x,y)); & \text{if } (x,y)\in J_k,\ k=0,\dots,m, \\ u(x_k^+,y) = u(x_k^-,y) + I_k(u(x_k^-,y)); & \text{if } y\in[0,b],\ k=1,\dots,m, \end{cases} \tag{2}$$

where $f\colon J\times E\to E$ and $I_k\colon E\to E$; $k=1,\dots,m$ are given continuous functions. Our considerations are based upon the Banach contraction principle and a fractional version of Gronwall's inequality. This paper is organized as follows. In Section 2, we collect some preliminary background needed in the following section. Our main result will be presented in Section 3. An illustrative example is presented in Section 4. This paper initiates the Ulam stability of the Darboux problem for fractional differential equations.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let E be a Banach space and let $J := [0,a]\times[0,b]$; $a,b>0$. Denote $L^1(J)$ the space of Bochner-integrable functions $u\colon J\to E$ with the norm

$$\|u\|_{L^1} = \int_0^a\int_0^b \|u(x,y)\|_E\,dydx,$$

where $\|.\|_E$ denotes a suitable complete norm on E.

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into E, and $\mathcal{C} := C(J)$ is the Banach space of all continuous functions from J into E with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(x,y)\in J}\|u(x,y)\|_E.$$

In all what follows consider the Banach space

$$PC = \big\{u\colon J\to E : u\in C(J_k);\ k=0,1,\dots,m, \text{ and there exist } u(x_k^-,y) \text{ and } u(x_k^+,y);\ k=1,\dots,m, \text{ with } u(x_k^-,y)=u(x_k,y) \text{ for each } y\in[0,b]\big\},$$

with the norm

$$\|u\|_{PC} = \sup_{(x,y)\in J}\|u(x,y)\|_E.$$

Definition 2.1 [35] Let $\theta=(0,0)$, $r_1,r_2\in(0,\infty)$ and $r=(r_1,r_2)$. For $f\in L^1(J)$, the expression

$$(I^r_\theta f)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)\,dtds,$$

is called the left-sided mixed Riemann–Liouville integral of order r, where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\xi)=\int_0^\infty t^{\xi-1}e^{-t}dt;\ \xi>0$.

In particular,

$$(I^\theta_\theta f)(x,y)=f(x,y),\ (I^\sigma_\theta f)(x,y)=\int_0^x\int_0^y f(s,t)\,dtds;\text{ for almost all }(x,y)\in J,$$

where $\sigma=(1,1)$. For instance, $I^r_\theta f$ exists for all $r_1,r_2\in(0,\infty)$, when $f\in L^1(J)$. Note also that when $f\in\mathcal{C}$, then $(I^r_\theta f)\in\mathcal{C}$, moreover

$$(I^r_\theta f)(x,0)=(I^r_\theta f)(0,y)=0;\ x\in[0,a],\ y\in[0,b].$$

Example 2.2 Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$, then

$$I^r_\theta x^\lambda y^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}x^{\lambda+r_1}y^{\omega+r_2};\ \text{ for almost all }(x,y)\in J.$$

By $1-r$ we mean $(1-r_1,1-r_2)\in[0,1)\times[0,1)$. Denote by $D^2_{xy}:=\frac{\partial^2}{\partial x\partial y}$, the mixed second order partial derivative.

Definition 2.3 [35] Let $r\in(0,1]\times(0,1]$ and $f\in L^1(J)$. The Caputo fractional-order derivative of order r of f is defined by the expression

$$\begin{aligned}{}^cD^r_\theta f(x,y)&=(I^{1-r}_\theta D^2_{xy}f)(x,y)\\&=\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\int_0^x\int_0^y\frac{D^2_{st}f(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\,dtds.\end{aligned}$$

The case $\sigma=(1,1)$ is included and we have

$$({}^cD^\sigma_\theta f)(x,y)=(D^2_{xy}f)(x,y);\ \text{for almost all }(x,y)\in J.$$

Example 2.4 Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$, then

$${}^cD^r_\theta x^\lambda y^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)}x^{\lambda-r_1}y^{\omega-r_2};\ \text{for almost all }(x,y)\in J.$$

Let $a_1\in[0,a]$, $z=(a_1,0)$, $J_z=(a_1,a]\times[0,b]$, $r_1,r_2>0$ and $r=(r_1,r_2)$. For $u\in L^1(J_z)$, the expression

$$(I^r_z u)(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{a_1^+}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)\,dtds,$$

is called the left-sided mixed Riemann–Liouville integral of order r of u.

Definition 2.5 [35]. For $u \in L^1(J_z)$ where $D^2_{xy}u$ is Bochner integrable on $[x_k, x_{k+1}] \times [0, b]$, $k = 0, \dots, m$, the Caputo fractional order derivative of order r of u is defined by the expression

$$({}^cD^r_z f)(x, y) = (I^{1-r}_z D^2_{xy} f)(x, y).$$

Now, we consider the Ulam stability of impulsive fractional differential equations (2). Let ϵ be a positive real number and $\Phi\colon J \to [0, \infty)$ be a continuous function. We consider the following inequalities

$$\begin{cases} \|{}^cD^r_{x_k} u(x, y) - f(x, y, u(x, y))\|_E \le \epsilon; \text{ if } (x, y) \in J_k,\ k = 0, \dots, m, \\ \|u(x^+_k, y) - u(x^-_k, y) - I_k(u(x^-_k, y))\|_E \le \epsilon; \text{ if } y \in [0, b],\ k = 1, \dots, m. \end{cases} \tag{3}$$

$$\begin{cases} \|{}^cD^r_{x_k} u(x, y) - f(x, y, u(x, y))\|_E \le \Phi(x, y); \text{ if } (x, y) \in J_k,\ k = 0, \dots, m, \\ \|u(x^+_k, y) - u(x^-_k, y) - I_k(u(x^-_k, y))\|_E \le \Phi(x, y); \text{ if } y \in [0, b],\ k = 1, \dots, m. \end{cases} \tag{4}$$

$$\begin{cases} \|{}^cD^r_{x_k} u(x, y) - f(x, y, u(x, y))\|_E \le \epsilon\Phi(x, y); \text{ if } (x, y) \in J_k,\ k = 0, \dots, m, \\ \|u(x^+_k, y) - u(x^-_k, y) - I_k(u(x^-_k, y))\|_E \le \epsilon\Phi(x, y); \text{ if } y \in [0, b],\ k = 1, \dots, m. \end{cases} \tag{5}$$

In the theory of functional differential equations, there are some special kind of data dependence, (see [29, 30, 41]). Following these results in mind, we shall present four types of Ulam stability for the problem (2).

Definition 2.6 Problem (2) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (3), there exists a solution $v \in PC$ Problem (2) with

$$\|u(x, y) - v(x, y)\|_E \le \epsilon c_{f,m};\ (x, y) \in J.$$

Definition 2.7 Problem (2) is generalized Ulam–Hyers stable if there exists $\Theta_{f,m} \in PC([0, \infty), [0, \infty))$, $\Theta_{f,m}(0) = 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (3) there exists a solution $v \in PC$ Problem (2) with

$$\|u(x, y) - v(x, y)\|_E \le \Theta_{f,m}(\epsilon);\ (x, y) \in J.$$

Definition 2.8 Problem (2) is Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{f,m,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in PC$ of the inequality (5) there exists a solution $v \in PC$ Problem (2) with

$$\|u(x, y) - v(x, y)\|_E \le \epsilon c_{f,m,\Phi}\Phi(x, y);\ (x, y) \in J.$$

Definition 2.9 Problem (2) is generalized Ulam–Hyers–Rassias stable with respect to Φ if there exists a real number $c_{f,m,\Phi} > 0$ such that for each solution $u \in PC$ of the inequality (4) there exists a solution $v \in PC$ Problem (2) with

$$\|u(x, y) - v(x, y)\|_E \le c_{f,m,\Phi}\Phi(x, y);\ (x, y) \in J.$$

Remark 2.10 It is clear that

(i) Definition 2.6 $\Rightarrow$ Definition 2.7,

(ii) Definition 2.8 $\Rightarrow$ Definition 2.9,

(iii) Definition 2.8 for $\Phi(x,y)=1 \ \Rightarrow$ Definition 2.6.

Remark 2.11 A function $u \in PC$ is a solution of the inequality (3) if and only if there exist a function $g \in PC$ and a sequence $g_k;\ k=1,\dots,m$ (which depend on u) such that

(i) $\|g(x,y)\|_E \leq \epsilon$ and $\|g_k\|_\infty \leq \epsilon;\ k=1,\dots,m,$

(ii) ${}^cD^r_{x_k}u(x,y)=f(x,y,u(x,y))+g(x,y);$ if $(x,y)\in J_k,\ k=0,\dots,m,$

(iii) $u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y))+g_k;$ if $y\in[0,b],\ k=1,\dots,m.$

One can have similar remarks for the inequalities (4) and (5). So, the Ulam stabilities of the impulsive fractional differential equations are some special types of data dependence of the solutions of impulsive fractional differential equations.

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 2.12 (Gronwall's lemma) [17] *Let $\upsilon\colon J\to[0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c>0$ and $0<r_1,r_2<1$ such that*

$$\upsilon(x,y)\leq\omega(x,y)+c\int_0^x\int_0^y\frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\,dtds,$$

then there exists a constant $\delta=\delta(r_1,r_2)$ such that

$$\upsilon(x,y)\leq\omega(x,y)+\delta c\int_0^x\int_0^y\frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}\,dtds,$$

for every $(x,y)\in J$.

3 Main results

In this section, we present conditions for the Ulam stability of problem (2).

We need the following auxiliary Lemma whose proof in when E is of a finite dimension was given in [6].

Lemma 3.1 [6] *Let $\mu(x,y)=\varphi(x)+\psi(y)-\varphi(0)$. A function $u\in PC$ is a solution of the fractional integral equations*

$$\begin{cases} u(x,y)=\mu(x,y)+\int_0^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u(s,t))\,dtds;\ (x,y)\in J_0,\\ u(x,y)=\mu(x,y)+\sum_{i=1}^k(I_i(u(x_i^-,y))-I_i(u(x_i^-,0)))\\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds\\ +\int_{x_k}^x\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(s,t,u(s,t))\,dtds;\ (x,y)\in J_k,\ \ k=1,\dots,m,\end{cases} \tag{6}$$

if and only if u is a solution of the problem (1).

Lemma 3.2 *If $u \in PC$ is a solution of the inequality (3) then u is a solution of the following integral inequality*

$$\begin{cases} \|u(x,y)-\mu(x,y)-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds\|_E \\ \leq \frac{\epsilon a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)};\ \text{if } (x,y)\in[0,x_1]\times[0,b], \\ \|u(x,y)-\mu(x,y)-\sum_{i=1}^k(I_i(u(x_i^-,y))-I_i(u(x_i^-,0))) \\ -\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds \\ -\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds\|_E \\ \leq \frac{2\epsilon a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)};\ \text{if } (x,y)\in(x_k,x_{k+1}]\times[0,b],\ k=1,\dots,m. \end{cases} \tag{7}$$

Proof by Remark 2.11 we have that

$$\begin{cases} {}^cD_{x_k}^r u(x,y)=f(x,y,u(x,y))+g(x,y);\ \text{ if } (x,y)\in J_k,\ k=0,\dots,m, \\ u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y))+g_k;\ \text{ if } y\in[0,b],\ k=1,\dots,m. \end{cases}$$

Then

$$u(x,y)=$$

$$=\begin{cases} \mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}(g(s)+f(s,t,u(s,t)))\,dtds; \\ \text{ if } (x,y)\in[0,x_1]\times[0,b], \\ \mu(x,y)+\sum_{i=1}^k(I_i(u(x_i^-,y))-I_i(u(x_i^-,0))) \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}(g(s)+f(s,t,u(s,t)))\,dtds \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}(g(s)+f(s,t,u(s,t)))\,dtds; \\ \text{ if } (x,y)\in(x_k,x_{k+1}]\times[0,b],\ k=1,\dots,m. \end{cases}$$

Thus, it follows that

$$\begin{cases} \|u(x,y)-\mu(x,y)-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds\|_E \\ =\|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dtds\|_E; \\ \text{ if } (x,y)\in[0,x_1]\times[0,b], \\ \|u(x,y)-\mu(x,y)-\sum_{i=1}^k(I_i(u(x_i^-,y))-I_i(u(x_i^-,0))) \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds \\ -\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds\|_E \\ =\|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^k\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dtds \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}g(s,t)\,dtds\|_E; \\ \text{ if } (x,y)\in(x_k,x_{k+1}]\times[0,b],\ k=1,\dots,m. \end{cases}$$

Hence, we obtain (7). □

Remark 3.3 We have similar results for the solutions of the inequalities (4) and (5).

In the sequel, we need the following theorem.

Theorem 3.4 *[6] Assume that the following hypotheses hold*

(H_1) *There exists a constant* $l_f > 0$ *such that*

$$\|f(x,y,u) - f(x,y,\overline{u})\|_E \le l_f\|u - \overline{u}\|_E,$$

for each $(x,y) \in J$*, and each* $u, \overline{u} \in E$*,*

(H_2) *There exists a constant* $l^* > 0$ *such that*

$$\|I_k(u) - I_k(\overline{u})\|_E \le l^*\|u - \overline{u}\|_E, \text{ for each } u, \overline{u} \in E, \ k = 1, \dots, m.$$

If

$$2ml^* + \frac{2l_f a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1, \tag{8}$$

then (1) *has a unique solution on* J*.*

Theorem 3.5 *Assume that assumptions* (H_1), (H_2) *and the following hypotheses hold*

(H_3) $\Phi \in L^1(J, [0,\infty))$ *and there exists* $\lambda_\Phi > 0$ *such that, for each* $(x,y) \in J$ *we have*

$$(I_\theta^r \Phi)(x,y) \le \lambda_\Phi \Phi(x,y),$$

(H_4) $\|I_k(u)\|_E \le \Phi(x,y)$; *for each* $u \in E$, $k = 1, \dots, m$.

If the condition (8) is satisfied, then problem (2) is generalized Ulam-Hyers-Rassias stable.

Proof Let $u \in PC$ be a solution of the inequality (4). By Theorem 3.4 there v is a unique solution of the problem (1). Then we have

$$v(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,v(s,t))dtds; \\ \text{ if } (x,y) \in [0,x_1] \times [0,b], \\ \mu(x,y) + \sum_{i=1}^k (I_i(v(x_i^-,y)) - I_i(v(x_i^-,0))) \\ \ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^k \int_{x_{i-1}}^{x_i} \int_0^y (x_i-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,v(s,t))dtds \\ \ + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,v(s,t))dtds; \\ \text{ if } (x,y) \in (x_k, x_{k+1}] \times [0,b], \ k = 1, \dots, m. \end{cases}$$

By the differential inequality (4) and (H_3), we obtain

$$\|u(x,y) - \mu(x,y) - \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s,t,u(s,t))dtds\|_E$$
$$\le (I_\theta^r \Phi)(x,y) \le \lambda_\Phi \Phi(x,y); \text{ if } (x,y) \in J_0,$$

and

$$\|u(x,y)-\mu(x,y)-\sum_{i=1}^{k}(I_i(u(x_i^-,y))-I_i(u(x_i^-,0)))$$
$$-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))dtds$$
$$-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))dtds\|_E$$
$$\leq\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}\Phi(s,t)dtds$$
$$+\int_{x_k}^{x}\int_0^y\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\Phi(s,t)dtds$$
$$\leq(1+k)(I_\theta^r\Phi)(x,y)\leq(1+k)\lambda_\Phi\Phi(x,y);\ \text{if } (x,y)\in J_k;\ k=1,\dots,m.$$

Thus, for each $(x,y)\in J_0$, we have

$$\|u(x,y)-v(x,y)\|_E\leq$$
$$\leq\|u(x,y)-\mu(x,y)-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds\|_E$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,u(s,t))-f(s,t,v(s,t))\|_E\,dtds$$
$$\leq\lambda_\Phi\Phi(x,y)+\frac{l_f}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t)-v(s,t)\|_E\,dtds,$$

and for each $(x,y)\in J_k$; $k=1,\dots,m$, we have

$$\|u(x,y)-v(x,y)\|_E\leq\|u(x,y)-\mu(x,y)-\sum_{i=1}^{k}(I_i(u(x_i^-,y))-I_i(u(x_i^-,0)))$$
$$-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds$$
$$-\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))\,dtds\|_E$$
$$+\sum_{i=1}^{k}\|I_i(u(x_i^-,y))-I_i(v(x_i^-,y))\|_E+\sum_{i=1}^{k}\|I_i(u(x_i^-,0))-I_i(v(x_i^-,0))\|_E$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\sum_{i=1}^{k}\int_{x_{i-1}}^{x_i}\int_0^y(x_i-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,u(s,t))-f(s,t,v(s,t))\|_E\,dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,u(s,t))-f(s,t,v(s,t))\|_E\,dtds$$

$$\leq (1+k)\lambda_\Phi\Phi(x,y)+4k\Phi(x,y)$$
$$+\frac{l_f(1+k)}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t)-v(s,t)\|_E\,dtds.$$

Hence, for each $(x,y)\in J_k$; $k=0,\dots,m$, we get

$$\|u(x,y)-v(x,y)\|_E\leq (1+k)\lambda_\Phi\Phi(x,y)+4k\Phi(x,y)$$
$$+\frac{l_f(1+k)}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t)-v(s,t)\|_E\,dtds.$$

From Lemma 2.12, there exists a constant $\delta=\delta(r_1,r_2)$ such that

$$\|u(x,y)-v(x,y)\|_E\leq[4k+(1+k)\lambda_\Phi]\Phi(x,y)$$
$$+\frac{\delta l_f(1+k)[4k+(1+k)\lambda_\Phi]}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}\Phi(s,t)\,dtds$$
$$\leq[4m+(1+m)\lambda_\Phi][1+(1+m)\delta l_f\lambda_\Phi]\Phi(x,y):=c_{f,m,\Phi}\Phi(x,y).$$

Finally, problem (2) is generalized Ulam–Hyers–Rassias stable.

4 An example

Let

$$E=l^1=\left\{w=(w_1,w_2,\dots,w_n,\dots)\colon\sum_{n=1}^{\infty}|w_n|<\infty\right\},$$

be the Banach space with norm $\|w\|_E=\sum_{n=1}^{\infty}|w_n|$. Consider the following infinite system of partial hyperbolic fractional impulsive differential equations of the form

$$\begin{cases}{}^cD^r_{x_k}u(x,y)=f(x,y,u(x,y));\ \text{ if } (x,y)\in J_k,\ k=0,\dots,m,\\ u(x_k^+,y)=u(x_k^-,y)+I_k(u(x_k^-,y));\ \text{ if } y\in[0,1],\ k=1,\dots,m,\end{cases}\tag{9}$$

where $J=[0,1]\times[0,1]$, $(r_1,r_2)\in(0,1]\times(0,1]$, $u=(u_1,u_2,\dots,u_n,\dots)$, $f=(f_1,f_2,\dots,f_n,\dots)$,

$${}^cD^r_{x_k}u=({}^cD^r_{x_k}u_1,{}^cD^r_{x_k}u_2,\dots,{}^cD^r_{x_k}u_n,\dots);\ k=1,\dots,m,$$

$$f_n(x,y,u)=\frac{1}{(10e^{x+y+4})(1+\|u_n\|_E)};\ (x,y)\in[0,1]\times[0,1],\ n\in\mathbb{N},$$

and

$$I_k(u(x_k^-,y))=\frac{xy^2}{(30e^{x+y+4})(1+\|u(x_k^-,y)\|_E)},\ y\in[0,1],\ k=1,\dots,m.$$

Clearly, the function f is continuous. For each $n\in\mathbb{N}$, $u,\overline{u}\in E$ and $(x,y)\in[0,1]\times[0,1]$, we have

$$\|f(x,y,u(x,y))-f(x,y,\overline{u}(x,y))\|_E\leq\frac{1}{10e^4}\|u-\overline{u}\|_E,$$

and

$$\|I_k(u) - I_k(\overline{u})\|_E \leq \frac{1}{30e^4}\|u - \overline{u}\|_E.$$

Then, the hypotheses (H_1) and (H_2) are satisfied with

$$l_f = \frac{1}{10e^4} \quad \text{and} \quad l^* = \frac{1}{30e^4}.$$

We shall show that condition (8) holds with $a = b = 1$. Indeed, $\Gamma(1 + r_i) > \frac{1}{2}$; $i = 1, 2$, and if we assume for instance that the number of impulses $m = 3$, then we get

$$2ml^* + \frac{2l_f a^{r_1} b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} = \frac{1}{e^4} + \frac{1}{5e^4\Gamma(1 + r_1)\Gamma(1 + r_2)} < \frac{9}{5e^4} < 1.$$

The hypothesis (H_3) is satisfied with $\Phi(x, y) = xy^2$ and $\lambda_\Phi = 8$. Indeed, a simple computation shows that

$$(I_\theta^r\Phi)(x, y) = \frac{\Gamma(2)\Gamma(3)}{\Gamma(2 + r_1)\Gamma(3 + r_2)} x^{1+r_1} y^{2+r_2} < 8xy^2 = \lambda_\Phi\Phi(x, y).$$

Finally, for each $(x, y) \in [0, 1] \times [0, 1]$ and $u \in E$, we have

$$\|I_k(u)\|_E \leq xy^2 = \Phi(x, y).$$

Hence, the hypothesis (H_4) is satisfied. Consequently Theorem 3.5 implies that the problem (9) is generalized Ulam–Hyers–Rassias stable.

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