Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **52**, 2 (2013) 71–83

Weak Consistency: A New Approach to Consistency in the Saaty's Analytic Hierarchy Process*

Věra JANDOVÁ [a], Jana TALAŠOVÁ [b]

Department of Mathematical Analysis and Applications of Mathematics
Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic
[a] *e-mail: vera.jandova@upol.cz*
[b] *e-mail: jana.talasova@upol.cz*

(Received May 31, 2013)

Abstract

In the decision making methods based on the pairwise comparison there is very important to enter the preferences of compared elements in the rational way. Only in this case we are able to obtain the reasonable solution. In the Analytic Hierarchy Process (AHP) there is set a strict consistency condition in order to keep the rationality of preference intensities between compared elements. But this requirement for the Saaty's matrix is not achievable in the real situations because of the Saaty's scale which is used in this method. That is why instead of the consistency condition we suggest a weak consistency condition which is very natural and more suitable for the linguistic descriptions of the Saaty's scale and as a result of it, it is easier to reach this requirement in the real situations. In addition, if we order compared elements from the most preferred to the least preferred, it is very easy to check if the weak consistency is satisfied. Big advantage of our approach to the consistency is that its satisfaction can be easily approved. It is also possible to control the weak consistency of the Saaty's matrix during the filling of the intensities of preferences. We also show on the example that there can be situations in which the weak consistency condition is more suitable for checking the rationality of the preferences than the Saaty's consistency ratio.

Key words: decision making, consistency, Saaty's AHP

2010 Mathematics Subject Classification: 62C86

*Supported by the grant PrF 2013 013 Mathematical models of the Internal Grant Agency of Palacky University in Olomouc.

1 Introduction

The Analytic Hierarchy Process (AHP) is based on the constructing pairwise comparison matrices of elements. First, the decision maker compares criteria with respect to the overall goal of the problem, in the next step alternatives with respect to each criterion. The Saaty's scale is used for setting these matrices. The weights of the criteria and the evaluations of the alternatives with respect to each criterion are obtained from these matrices. Finally, the aggregation with using the weighted average is applied for gaining the evaluation of each alternative with respect to the overall goal.

Creating a hierarchy structure of the decision making model, computing the weights and the evaluations and the final synthesis of the evaluations are described in [8]. In this section we will be concerned with the setting the pairwise comparison matrix. Let us consider criteria $C_1, C_2, \dots, C_n$ and their respective weights $w_1, w_2, \dots w_n$. We want to create the *Saaty's matrix* $S = \{s_{ij}\}_{i,j}^n$ where $s_{ij} \approx \frac{w_i}{w_j}$. I.e. s_{ij} corresponds to preference intensity of criterion C_i to criterion C_j. For creating the matrix S we will use *fundamental Saaty's scale* which was proposed by Saaty [7] on the basis that human brain is able to distinguish only 9 levels of preferences. The scale is described in Table 7.

Preference intensity	Linguistic meaning
1	First object is *equally important as* the second one
3	First object is *moderately more important than* the second one
5	First object is *strongly more important than* the second one
7	First object is *very strongly more important than* the second one
9	First object is *extremely more important than* the second one
$2, 4, 6, 8$	Intermediete values between the two adjacent judgements

Table 7: The fundamental Saaty's scale

If C_i is preferred to C_j, s_{ij} is equal to one of the values from Table 7 and it can be interpreted as: C_i is s_{ij}-times more important than C_j. Then C_j creates $\frac{1}{s_{ij}}$ of the importance of the criterion C_i. It means that the matrix S must be *reciprocal*, i.e. $s_{ji} = \frac{1}{s_{ij}}$ for all $i, j = 1, 2, \dots, n$. Further, it is obvious that $s_{ii} = 1$ for all $i = 1, 2, \dots, n$. Now let us summarize these attributes in the following definiton.

Definition 1 Let $S = \{s_{ij}\}_{i,j=1}^n$ be a square matrix. We say that S is *Saaty's matrix* if the following holds for all $i, j \in \{1, 2, \dots, n\}$:

$$s_{ii} = 1, \quad s_{ji} = \frac{1}{s_{ij}}, \quad s_{ij} \in \left\{ \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9 \right\}.$$

In order to obtain a correct vector of the relative preferences of criteria from the Saaty's matrix, the given preference intensities must be entered in the reasonable way. This means the matrix of prefence intensities should be *consistent*, i.e. for all $i, j, k = 1, 2, \ldots, n$ must hold

$$s_{ij} = s_{ik} s_{kj}. \tag{1.1}$$

However, this requirement is not fully achievable for the Saaty's matrix in the real situations. Consider one element moderately more important than the second one and the second one strongly more important than the third one. According to the condition (1.1), the first element would have to be 15-times more important than the third one. But this value is not included in the Saaty's scale. Thus, Saaty [10] defined a *consistency index* (CI) based on the *principal eigenvalue* $\lambda_{\max}$ of the Saaty's matrix S

$$CI = \frac{\lambda_{\max} - n}{n - 1}.$$

According to the Perron-Frobenius theorem (see [6], page 673), principal eigenvalue $\lambda_{\max}$ always exists for the Saaty's matrix and it holds $\lambda_{\max} \geq n$; for fully consistent matrix $\lambda_{\max} = n$ (see [10]). It means $CI \geq 0$ and the less value the less inconsistency. For measuring the level of consistency Saaty [10] proposed a *consistency ratio* (CR) which compares consistency index CI of the given matrix S and the *random index* (RI)

$$CR = \frac{CI}{RI(n)}.$$

$RI(n)$ represents the average consistency index of randomly generated Saaty's matrices of the dimension n, and a few first values (for more see [11]) can be found in Table 8.

n	3	4	5	6	7	8	9	10	11
RI(n)	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49	1.52

Table 8: Random index $RI(n)$

Saaty [10] established the matrix S has acceptable level of inconsistency and is considered consistent enough if $CR \leq 0.1$.

As was described above, for the fully consistent matrix S holds $\lambda_{\max} = n$ and $s_{ij} = \frac{w_i}{w_j}$ for all $i, j = 1, 2, \ldots, n$ (in general for the Saaty's matrix holds $\lambda_{\max} \geq n$ and $s_{ij} \approx \frac{w_i}{w_j}$). Thus, the vector of weights $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ of the criteria can be obtained as the eigenvector of S associated with the principal eigenvalue $\lambda_{\max}$ (see [9])

$$S\mathbf{w} = \lambda_{\max}\mathbf{w}. \tag{1.2}$$

The vector of the weights can be derived also with using the Logarithmic Least Squares Method (see [4]). Then the weights are represented by the geometric means of the rows of the matrix S. For all $i = 1, 2, \ldots, n$ holds

$$w_i = \sqrt[n]{\prod_{j=1}^{n} s_{ij}}. \tag{1.3}$$

The weights (1.2) and (1.3) can be transformed into the normalized weights $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ with using the relation

$$v_i = \frac{w_i}{\sum\limits_{j=1}^{n} w_j}$$

for all $i = 1, 2, \ldots, n$. The normalized weights obtained from (1.2) and (1.3) are equal for the fully consistent matrices. However, these two vectors differ when the full consistency is violated (see [3]).

As it was mentioned above, the Saaty's matrix S must be consistent if we want to derive a correct vector of weights. Because the consistency condition (1.1) is not fully achievable, beside Saaty also various authors tried to construct alternative measures of the consistency of the Saaty's pairwise comparison matrix. Alonso and Lamata [2] compute also a consistency ratio but instead of RI they use $RI^* = \frac{\lambda^*_{\max} - n}{n-1}$, where $\lambda^*_{\max}$ is the average value of the principal eigenvalue of randomly generated Saaty's matrices of the dimension n. Another approach is represented by Lamata and Pelaez [5] who suggest the consistency index CI^* using determinant and subdeterminants of the matrix S. For randomly generated matrices they determine the p-value and critical value CR^* which is used for making a decision if S is consistent enough.

The disadvantage of all of these methods is that it is not possible to control whether the matrix is consistent enough during inputting the preference intensities. Consistency is checked after the matrix is completed and if the level of consistency is not suitable, the decision maker usually must create a new matrix of pairwise comparisons. But it does not guarantee that the new matrix will be consistent enough. In the next section we are going to define our consistency condition which can be reached in the real situations and is possible to check its fulfilling during inserting the data.

2 The notion of the weak consistency

In this section we are going to introduce a new approach to the consistency requirement in the AHP method. This condition must retain the transitivity of the preference intensities, must be achievable in the real situations and must be controllable during filling the preference intensities. We want to define a condition which will help us to find the main inconsistencies in the decision making process and which fulfillment will lead to the acceptable vector of weights computed by the relation (1.3).

The Saaty's condition that $CR \leq 0.1$ operates with the principal eigenvalue $\lambda_{\max}$ of the matrix S and suggests whether the weights computed by (1.2) give us the acceptable solution. We want to define the consistency condition also for the method of the geometric means of the rows of the matrix S which is given by the formula (1.3). Look at the Saaty's matrix S. The columns of S can be interpreted as the repeated measurements of the relative preferences of n criteria. All the criteria are compared with the first one, then with the second one and so on until the n^{th} one. From the point of view of the mathematical statistics, these are *compositional data*, i.e. data bearing only relative information (see [1]). Information contained in this data can be expressed by estimating its mean value. A proper estimator of the mean value of this kind of data is a vector whose components are geometric means of the corresponding components of vectors representing single measurements (see [4]). This leads us back to the geometric means of the rows of the Saaty's matrix. In this respect, we are going to define the weak consistency condition. If the decision maker satisfies this condition throughout the process of inputting preference intensities, we can expect the individual measurements and the estimate of the mean value of the compositions will be better.

Thus, the weak consistency condition must be defined in the way which is easily achievable in the real situations and which is possible to control during the process of inputting data. This condition must also retain the transitivity of preferences like the consistency condition (1.1). But as we mentioned in the previous section, the condition (1.1) is not achievable in the real situations. If object A is strongly preferred ($s_{AB} = 5$) to B and B is very strongly preferred ($s_{BC} = 7$) to C, we need A is 35-times more preferred to C. But this number does not belong to the Saaty's scale. In addition, from two moderate preferences ($s_{AB} = 3$ and $s_{BC} = 3$) we should derive extreme preference ($s_{AC} = 9$) which intuitively seems to be too strong preference between the first and the third element. On account of this, we can define the weak consistency condition which is more suitable for the linguistic descriptions of the elements of the Saaty's scale and which helps us to find the principal inconsistecies in the pairwise comparisons. We require the result of the composition of two preference intensities is at least the biggest of them. Furthermore, if one element is equally preferred to the second one and the second one is more preferred to the third one, it is reasonable to expect that between the first and the third element there is larger preference of those two. If we return to our example given at the beginning of this paragraph, we would expect that A is at least 7-times more preferred to C and that two moderate preferences result in at least moderate preference. This approach is summarized in Definition 2.

Definition 2 Let $S = \{s_{ij}\}_{i,j=1}^{n}$ be the Saaty's matrix. We say that S is *weakly consistent* if the following holds for all $i, j, k \in \{1, 2, \dots, n\}$

$$s_{ij} > 1 \wedge s_{jk} > 1 \quad \Longrightarrow \quad s_{ik} \geq \max\{s_{ij}, s_{jk}\}; \tag{2.1}$$

$$(s_{ij} = 1 \wedge s_{jk} \geq 1) \vee (s_{ij} \geq 1 \wedge s_{jk} = 1) \quad \Longrightarrow \quad s_{ik} = \max\{s_{ij}, s_{jk}\}. \tag{2.2}$$

Remark 1 It can be easily seen that the weak consistency keeps the transitivity of the preferences. From the Definition 2 the following holds for all $i, j, k = 1, 2, \ldots, n$

$$s_{ij} > 1 \wedge s_{jk} > 1 \quad \Longrightarrow \quad s_{ik} \geq \max\{s_{ij}, s_{jk}\} > 1.$$

Because of the reciprocity, it would be possible to define the weak consistency equivalently with the elements less than or equal to 1 and with the minimum operator which is demonstrated in Theorem 1.

Theorem 1 *Let $S = \{s_{ij}\}_{i,j=1}^{n}$ be the Saaty's matrix. Then S is weakly consistent if and only if the following holds for all $i, j, k \in \{1, 2, \ldots, n\}$:*

$$s_{ij} < 1 \wedge s_{jk} < 1 \quad \Longrightarrow \quad s_{ik} \leq \min\{s_{ij}, s_{jk}\}; \tag{2.3}$$

$$(s_{ij} = 1 \wedge s_{jk} \leq 1) \vee (s_{ij} \leq 1 \wedge s_{jk} = 1) \quad \Longrightarrow \quad s_{ik} = \min\{s_{ij}, s_{jk}\}. \tag{2.4}$$

Proof

1. First, we prove that the weak consistency implies conditions (2.3) and (2.4):

   a) Let $s_{ij} < 1$ and $s_{jk} < 1$ for $i, j, k \in \{1, 2, \ldots, n\}$. Then from the reciprocity of S we get $s_{ji} > 1$ and $s_{kj} > 1$. The weak consistency implies that $s_{ki} \geq \max\{s_{ji}, s_{kj}\}$, i.e. $s_{ik} \leq \frac{1}{\max\{s_{kj}, s_{ji}\}}$. Hence, $s_{ik} \leq \frac{1}{s_{kj}} = s_{jk}$ and $s_{ik} \leq \frac{1}{s_{ji}} = s_{ij}$. In other words $s_{ik} \leq \min\{s_{jk}, s_{ij}\}$.

   b) Let $s_{ij} = 1$ and $s_{jk} \leq 1$ for $i, j, k \in \{1, 2, \ldots, n\}$. Then $s_{ji} = 1$ and $s_{kj} \geq 1$. The weak consistency implies that $s_{ki} = s_{kj}$. From the reciprocity $s_{ik} = s_{jk} = \min\{s_{ij}, s_{jk}\}$.

   c) Let $s_{ij} \leq 1$ and $s_{jk} = 1$ for $i, j, k \in \{1, 2, \ldots, n\}$. Then $s_{ji} \geq 1$ and $s_{kj} = 1$. The weak consistency implies $s_{ki} = s_{ji}$. From the reciprocity $s_{ik} = s_{ij} = \min\{s_{ij}, s_{jk}\}$.

2. Now let us suppose that S fulfills (2.3) and (2.4). We will prove that such matrix
   S is weakly consistent:

   a) Let $s_{ji} > 1$ and $s_{kj} > 1$ for $i, j, k \in \{1, 2, \ldots, n\}$. The reciprocity implies $s_{ij} < 1$ and $s_{jk} < 1$. From (2.3) we obtain $s_{ik} \leq \min\{s_{ij}, s_{jk}\}$. Then $s_{ik} \leq s_{ij}$ and $s_{ik} \leq s_{jk}$. From the reciprocity we get $s_{ki} \geq s_{ji}$ and $s_{ki} \geq s_{kj}$, i.e. $s_{ki} \geq \max\{s_{kj}, s_{ji}\}$.

   b) Let $s_{ji} = 1$ and $s_{kj} \geq 1$ for $i, j, k \in \{1, 2, \ldots, n\}$. The reciprocity implies $s_{ij} = 1$ and $s_{jk} \leq 1$. From (2.4) we obtain $s_{ik} = s_{jk} = \min\{s_{ij}, s_{jk}\}$. Thus, from reciprocity $s_{ki} = s_{kj} = \max\{s_{kj}, s_{ji}\}$.

   c) Let $s_{ji} \geq 1$ and $s_{kj} = 1$ for $i, j, k \in \{1, 2, \ldots, n\}$. The reciprocity implies $s_{ij} \leq 1$ and $s_{jk} = 1$. From (2.4) we obtain $s_{ik} = s_{ij} = \min\{s_{ij}, s_{jk}\}$. Thus, from the reciprocity $s_{ki} = s_{ji} = \max\{s_{kj}, s_{ji}\}$. □

The relations between the elements greater than 1 and lower than 1 can be described by Theorem 2 and Corollary 1. They give us also sufficient conditions for the Saaty's matrix S not to be weakly consistent.

Theorem 2 *Let $S = \{s_{ij}\}_{i,j=1}^{n}$ be a weakly consistent Saaty's matrix. If $s_{ij} > 1$ and $s_{jk} < 1$ for $i, j, k \in \{1, 2, \ldots, n\}$, then the following holds for s_{ik}:*

$$1 < s_{ik} \le s_{ij}, \qquad \text{if } s_{ij} > \frac{1}{s_{jk}} = s_{kj}; \tag{2.5}$$

$$1 > s_{ik} \ge s_{jk}, \qquad \text{if } s_{ij} < s_{kj}; \tag{2.6}$$

$$s_{ji} \le s_{ik} \le s_{ij}, \qquad \text{if } s_{ij} = s_{kj}. \tag{2.7}$$

Proof Let S be weakly consistent and let us consider $s_{ij} > 1$ and $s_{jk} < 1$ for $i, j, k \in \{1, 2, \ldots, n\}$. Then three possible relations between these elements can occurr:

1. Let us consider $s_{ij} > s_{kj}$.
   a) Let us suppose that $s_{ik} < 1$. The reciprocity then implies $s_{ki} > 1$. From the weak consistency we obtain

   $$(s_{ki} > 1 \wedge s_{ij} > 1) \quad \Longrightarrow \quad s_{kj} \ge \max\{s_{ij}, s_{ki}\},$$

   which is a contradiction to the assumption that $s_{ij} > s_{kj}$.

   b) Let us suppose that $s_{ik} = 1$. As $s_{kj} > 1$, we obtain from the weak consistency that $s_{ij} = \max\{s_{kj}, s_{ik}\} = s_{kj}$, which is again a contradiction to the assumption that $s_{ij} > s_{kj}$.

   c) Consequently, $s_{ik} > 1$ must hold. As $s_{kj} > 1$, the weak consistency implies that $s_{ij} \ge \max\{s_{ik}, s_{kj}\}$. Thus $s_{ij} \ge s_{ik} > 1$ holds.

2. Now let $s_{ij} < s_{kj}$.
   a) Let $s_{ik} > 1$. As in 1c) we obtain $s_{ij} \ge s_{kj}$, which is a contradiction to the assumption that $s_{ij} < s_{kj}$.

   b) Let $s_{ik} = 1$. As in 1b) we obtain $s_{ij} = s_{kj}$, which is again a contradiction to the assumption that $s_{ij} < s_{kj}$.

   c) Consequently, $s_{ik} < 1$ must hold. Analogically to 1a), we obtain $1 > s_{ik} \ge s_{jk}$.

3. Let $s_{ij} = s_{kj}$. As S is weakly consistent, one of the following situations may occur:
   a) Let $s_{ik} > 1$. Then as $s_{kj} > 1$, we obtain from the weak consistency $s_{ij} \ge \max\{s_{ik}, s_{kj}\}$. As $s_{ij} = s_{kj}$, to fulfill the implication (2.1) it has to hold that $s_{ij} \ge s_{ik} > 1$.

   b) Now let $s_{ik} < 1$. Then $s_{ki} > 1$ and as $s_{ij} > 1$, the weak consistency implies $s_{kj} \ge \max\{s_{ij}, s_{ki}\}$. As $s_{ij} = s_{kj}$, to fulfill the implication (2.1) it has to hold that $s_{ij} \ge s_{ki}$, i.e. $s_{ji} \le s_{ik} < 1$.

   c) The last situation we need to deal with is $s_{ik} = 1$. As $s_{kj} > 1$, the weak consistency implies that $s_{ij} = s_{kj}$. As this equation holds, a situation when $s_{ik} = 1$ can occur.

When we put together 3a)–3c), we obtain $s_{ji} \le s_{ik} \le s_{ij}$. □

Corollary 1 *Let $S = \{s_{ij}\}_{i,j=1}^n$ be a weakly consistent Saaty's matrix. If $s_{ij} < 1$ and $s_{jk} > 1$ for $i, j \in \{1, 2, \dots, n\}$, then the following holds for s_{ik}:*

$$1 < s_{ik} \le s_{jk}, \quad \text{if } s_{jk} > \frac{1}{s_{ij}} = s_{ji}; \tag{2.8}$$

$$s_{ij} \le s_{ik} < 1, \quad \text{if } s_{jk} < s_{ji}; \tag{2.9}$$

$$s_{kj} \le s_{ik} \le s_{jk}, \quad \text{if } s_{jk} = s_{ji}. \tag{2.10}$$

Proof The proof is analogical to the proof of Theorem 2—to obtain (2.8), (2.9) and (2.10), we again investigate three cases: $s_{jk} > s_{ji}$, $s_{jk} < s_{ji}$ and $s_{jk} = s_{ji}$ and for each of them we examine consequences of $s_{ik} > 1$, $s_{ik} < 1$ and $s_{ik} = 1$. □

Before inputting the preference intensities we can order the elements according to their importance from the most important to the least important. It is possible to perform it by the Pairwise Comparison Method (see [12, 13]). After the ordering we obtain $s_{ij} \ge 1$ for all $i, j = 1, 2, \dots, n$ such that $i < j$. Afterwards the upper triangle of the matrix S consists only from numbers $\{1, 2, \dots, 9\}$. It means that we control the fulfilment of the weak consistency only for this upper triangle. In this case it is very easy to verify the requirement of the weak consistency even during the process of entering data. The weak consistency means that the sequence of numbers must be nondecreasing in every row of the upper triangle and nonincreasing in every column of the upper triangle of S. Moreover, if $s_{ij} = 1$, $i \ne j$, then rows i and j must be equal and also columns i and j must be equal. These properties are summarized in Theorem 3.

Theorem 3 *Let $S = \{s_{ij}\}_{i,j=1}^n$ be the Saaty's matrix and let $s_{ij} \ge 1$ for all $i, j = 1, 2, \dots, n$ such that $i < j$. Then S is weakly consistent if and only if the following requirements hold for the upper triangle of S:*

1. *the sequence $\{s_{ij}\}_{j=i+1}^n$ is nondecreasing for all $i \in \{1, 2 \dots, n-1\}$;*
2. *the sequence $\{s_{ij}\}_{i=1}^{j-1}$ is nonincreasing for all $j \in \{2, 3, \dots, n\}$;*
3. *if $s_{ij} = 1$ then $s_{li} = s_{lj}$ for all $i, j, l \in \{1, 2, \dots, n\}$ such that $l < i < j$;*
4. *if $s_{ij} = 1$ then $s_{ik} = s_{jk}$ for all $i, j, k \in \{1, 2, \dots, n\}$ such that $i < j < k$.*

Proof Assume that $s_{ij} \ge 1$ for all $i, j = 1, 2, \dots, n$ such that $i < j$. Then the upper triangle of S consists only of numbers greater than 1. The lower triangle consists of reciprocal values and can contain number 1. The weak consistency is defined for all $s_{ij} \ge 1$ but in the proving that Definition 2 is in this case equivalent to the requirements 1–4 we do not need to concern with the consequences of $s_{ji} = 1$ where $i < j$ because of the following: if $s_{ij} = 1$ for $i < j$, then according to (2.2) it is $s_{ik} = s_{jk}$ for $i < j < k$ where $i, j, k = 1, 2, \dots, n$. Thanks to the reciprocity $s_{ij} = 1$ is equivalent to the $s_{ji} = 1$. Again from the condition (2.2) we obtain $s_{jk} = s_{ik}$. The relation $s_{ji} = 1$ gives us the same result as $s_{ij} = 1$ and because of this, it is sufficient to prove that requirements 1–4 are equivalent to the conditions (2.1) and (2.2) only for $i < j < k$.

1. Let $s_{ij} \neq 1$ for all $i < j$. First, we prove that if we have the Saaty's matrix without number 1 elsewhere than on the main diagonal, the requirements 1 and 2 are equivalent to (2.1).

   a) Let us suppose that the sequence $\{s_{ij}\}_{j=i+1}^{n}$ is nondecreasing for all $i \in \{1, 2 \ldots, n-1\}$. It can be rewritten as follows:

   $$s_{ii+1} \leq s_{ii+2} \leq \cdots \leq s_{in-1} \leq s_{in}$$

   for all $i \in \{1, 2 \ldots, n-1\}$. In other words, it means $s_{ij} \leq s_{ik}$ for all $i, j, k \in \{1, 2, \ldots, n\}$ such that $i < j < k$.

   b) Let us suppose that the sequence $\{s_{ik}\}_{i=1}^{k-1}$ is nonincreasing for all $k \in \{2, 3, \ldots, n\}$. It can be rewritten as follows:

   $$s_{1k} \geq s_{2k} \geq \cdots \geq s_{k-2k} \geq s_{k-1k}$$

   for all $k \in \{2, 3, \ldots, n\}$. In other words, it means $s_{ik} \geq s_{jk}$ for all $i, j, k \in \{1, 2, \ldots, n\}$ such that $i < j < k$.

   c) From a) and b) we obtain that the requirements 1 and 2 have the same meaning as $s_{ik} \geq s_{ij} \wedge s_{ik} \geq s_{jk}$ for all $i, j, k \in \{1, 2, \ldots, n\}$ such that $i < j < k$. Thus, $s_{ik} \geq \max\{s_{ij}, s_{jk}\}$ for all $i, j, k \in \{1, 2, \ldots, n\}$ such that $i < j < k$.

   We proved that if $s_{ij} \neq 1$ for all $i < j$, the requirements 1 and 2 are equivalent to (2.1).

2. Let $s_{ij} = 1$ for $i < j$. Now we prove that if the matrix S contains number 1 elswhere than on the main diagonal, the requirements 3 and 4 are equivalent to (2.2).

   a) Let $s_{ik} = s_{jk}$ for $i < j < k$. It is equivalent to $s_{ik} = \max\{1, s_{jk}\} = \max\{s_{ij}, s_{jk}\}$.

   b) Let $s_{li} = s_{lj}$ for $l < i < j$. It is equivalent to $s_{lj} = \max\{1, s_{ij}\} = \max\{s_{li}, s_{ij}\}$.

   It means that (2.2) is equivalent to the requirements 3 and 4.

3. Finally, we have to show that for the matrix S which contains number 1 elsewhere than on the main diagonal, the requirements 1–4 are equivalent to the conditions (2.1) and (2.2). In the first part of the proof we demonstrated that for the matrix with $s_{ij} \neq 1$ for $i \neq j$ the weak consistency is equivalent to the conditions 1 and 2. For the matrix where $s_{ij} = 1$ for $i \neq j$ holds the following: According to what was shown in the first part of the proof, it is obvious that (2.1) and (2.2) imply 1 and 2. If in some cases in the relation $s_{ik} \geq \max\{s_{ij}, s_{jk}\}$ we have equality, we still obtain nondecreasing sequences in rows and nonincreasing sequences in columns. The opposite implication does not hold because these sequences do not guarantee that the i^{th} row (or column) contains the same values as j^{th} row (or column). However, if we subjoin the conditions 3 and 4 then the requirements 1–4 imply (2.1) and (2.2). And because 3 and 4 are equivalent to (2.2), we have that 1–4 are equivalent to (2.1) and (2.2). □

The properties demonstrated in Theorem 3 can be investigated in whole matrix S instead of the upper triangle. This is summarized in Consequence 1.

Consequence 1 *Let S be the Saaty's matrix and let $s_{ij} \geq 1$ for all $i, j = 1, 2, \ldots, n$ such that $i < j$. Then S is weakly consistent if and only if the following requirements hold:*

1. *the sequence $\{s_{ij}\}_{j=1}^{n}$ is nondecreasing for all $i \in \{1, 2 \ldots, n\}$;*
2. *if $s_{ij} = 1$ for $i, j \in \{1, 2, \ldots, n\}$ such that $i \neq j$ then $s_{li} = s_{lj}$ for all $l = 1, 2, \ldots, n$.*

Proof The equivalency results from Theorem 3 and the reciprocity of the Saaty's matrix S. The requirement 1 can be obtained from property 1 and the reciprocal property of 2 in Theorem 3. The requirement 2 consists of the property 3 and the reciprocal property of 4 in Theorem 3. □

Example 1

1. The following Saaty's matrix is weakly consistent according to Theorem 3.

$$\begin{pmatrix} 1 & 3 & 5 & 5 & 9 \\ \frac{1}{3} & 1 & 5 & 5 & 7 \\ \frac{1}{5} & \frac{1}{5} & 1 & 1 & 3 \\ \frac{1}{5} & \frac{1}{5} & 1 & 1 & 3 \\ \frac{1}{9} & \frac{1}{7} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

Every row in the upper triangle creates nondecreasing sequence and every column in the upper triangle generates nonincreasing sequence. Moreover, the 3$^{\text{rd}}$ and 4$^{\text{th}}$ rows contain equal values. The same holds for 3$^{\text{rd}}$ and 4$^{\text{th}}$ columns. This condition also must be satisfied because there is number 1 on the position $(3, 4)$.

2. On the other hand, the following Saaty's matrix is not weakly consistent according to Theorem 3.

$$\begin{pmatrix} 1 & 3 & 5 & 5 & 9 \\ \frac{1}{3} & 1 & 5 & 8 & 7 \\ \frac{1}{5} & \frac{1}{5} & 1 & 2 & 2 \\ \frac{1}{5} & \frac{1}{8} & \frac{1}{2} & 1 & 2 \\ \frac{1}{9} & \frac{1}{7} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

a) The second row of the upper triangle of the matrix contains the sequence $5, 8, 7$ which is not nondecreasing. The nonincreasing sequence in the fourth column of the upper triangle is also violated: $5, 8, 2$. Failure of the condition from the definition is possible to observe for example here: from $s_{13} = 5$ and $s_{34} = 8$ we should obtain s_{14} is at least 8, but $s_{14} = 5$.

b) Now suppose the value on the position $(2, 4)$ is empty. We want to input it in order to the matrix is weakly consistent. The second row indicates that number should belong to the set $\{5, 6, 7\}$ and the fourth column denotes it belongs to the set $\{2, 3, 4, 5\}$. To keep the weak consistency the only choice is number 5.

The concept of the weak consistency (2.1), (2.2) represents a weakening of the concept of the consistency (1.1). This demonstrates the following theorem.

Theorem 4 *Let* $S = \{s_{ij}\}_{i,j=1}^{n}$ *be consistent Saaty's matrix, i.e.* $s_{ik} = s_{ij}s_{jk}$ *for all* $i, j, k = 1, 2, \ldots, n$. *Then* S *is also weakly consistent.*

Proof Let S be consistent, i.e. (1.1) is fulfilled. Then $s_{ij} > 1$ and $s_{jk} > 1$ imply $s_{ik} = s_{ij}s_{jk} > \max\{s_{ij}, s_{jk}\}$, which means that the first condition of the weak consistency (2.1) is satisfied. Next if $s_{ij} = 1$, then $s_{ik} = s_{jk} = \max\{s_{ij}, s_{jk}\}$ and if $s_{jk} = 1$, then $s_{ik} = s_{ij} = \max\{s_{ij}, s_{jk}\}$. The second condition of the weak consistency (2.2) is also fulfilled. □

The implication in Theorem 4 holds only for the consistency defined by (1.1). The similar implication we do not obtain for matrix considered consistent enough according to the Saaty's consistency ratio CR. The condition that $CR \leq 0.1$ can be reached also for matrix, where the decision maker was not able to keep the preference ordering of the elements—at some place he prefers element A to element B and at the same time he enters information that B is preferred to A. In this respect, the weak consistency proposed by Definition 2 seems to be a better weakening of the consistency condition (1.1) than the requirement that the consistency ratio must be less or equal to 0.1. This situation will be illustrated in the first part of Example 2.

Example 2

1. Let us consider the following Saaty's matrix. It will be demonstrated that this matrix is consistent enough according to the consistency ratio and at the same time it is not weakly consistent.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{2} & 1 & 3 & 2 \\ \frac{1}{3} & \frac{1}{3} & 1 & 2 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

Its principal eigenvalue is $\lambda_{\max} = 4.1179$. The consistency index $CI = \frac{4.1179-4}{4-1} = 0.0393$ and the consistency ratio $CR = \frac{0.0393}{0.89} = 0.0442 < 0.1$. According to the Saaty's condition, this matrix is considered consistent enough, i.e. it has acceptable level of the inconsistency. However, we can observe that the decision maker did not keep the preference ordering. The second row suggests that the fourth element is preferred to the third one, while the first and the third row demonstrates that the third element is preferred to the fourth one. In spite of this contradiction, Saaty considers this matrix consistent enough. On the other hand, it can be easily obtained that the matrix is not weakly consistent because the sequence $3, 2$ in the second row of the upper triangle of S is not nondecreasing. It is also possible to examine the condition from the definiton: for $s_{23} = 3$ and $s_{34} = 2$ we would expect $s_{24} \geq \max\{s_{23}, s_{34}\} = 3$. However, $s_{24} = 2$ violates the condition (2.1) of the weak consistency.

2. The following Saaty's matrix is not consistent enough according to the condition $CR \leq 0.1$ but is weakly consistent.

$$\begin{pmatrix} 1 & 3 & 5 & 5 & 7 & 9 \\ \frac{1}{3} & 1 & 3 & 3 & 5 & 7 \\ \frac{1}{5} & \frac{1}{3} & 1 & 3 & 5 & 7 \\ \frac{1}{5} & \frac{1}{3} & \frac{1}{3} & 1 & 5 & 7 \\ \frac{1}{7} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 1 & 5 \\ \frac{1}{9} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{5} & 1 \end{pmatrix}$$

Its principal eigenvalue is $\lambda_{\max} = 6.7345$. The consistency index $CI = \frac{6.7345-6}{6-1} = 0.1469$ and the consistency ratio $CR = \frac{0.1469}{1.25} = 0.1175 > 0.1$. According to the Saaty's condition, this matrix is not considered consistent enough. On the other hand, we can see that the matrix is weakly consistent: in the upper triangle there are only numbers greater than 1, every row creates nondecreasing sequence and every column represents nonincreasing sequence. Although the matrix is not consistent enough according to Saaty, the weak consistency guarantees the transitivity of the prefences and informs us that two preference intensities result in at least the same preference intensity as the bigger of the previous two. This gives us the information that the decision maker entered the prefences and their intensities in the rational way.

3 Conclusion

The new approach to the consistency condition for the Saaty's matrix was introduced. The weak consistency was suggested to help us to find the main inconsistencies in the data, to be easily verified, to be possible to control it during filling the preference intensities and to be more suitable for linguistic meanings of the elements of the Saaty's scale.

The weak consistency was defined only for the pair of elements greater than or equal to 1. The properties for the other pairs of elements were demonstrated as a result of the reciprocity of the matrix. For the matrix where the elements are ordered from the most preferred to the least preferred, it was shown that it is very easy to monitor the achieving of this condition during inputting the preference intensities. Further it was demonstrated that the weak consistency condition guarantees the transitivity of the preferences and in this sense it is more suitable for detecting the rationality of the preferences than the Saaty's consistency ratio.

References

[1] Aitchison, J.: *The Statistical Analysis of Compositional Data.* Journal of the Royal Statistical Society. Series B (Methodological) **44**, 2 (1982), 139–177.

[2] Alonso, J. A., Lamata, M. T.: *Consistency in the Analytic Hierarchy Process: A New Approach.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **14**, 4 (2006), 445–459.

[3] Budescu, D. V., Zwick, R., Rapoport, A.: *A comparison of the eigenvalue method and the geometric mean procedure for ratio scaling.* Applied Psychological Measurement **10**, 1 (1986), 69–78.

[4] Crawford, G., Williams, C.: The Analysis of Subjective Judgment Matrices. *The Rand Corporation*, California, 1985.

[5] Lamata, M. T., Pelaez, J. I.: *A Method for Improving the Consistency Judgements.* International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems **10**, 6 (2002), 677–686.

[6] Meyer, C. D.: Matrix Analysis and Applied Linear Algebra Book and Solutions Manual. *SIAM: Society for Industrial and Applied Mathematics*, Philadelphia, 2000.

[7] Saaty, T. L.: *Axiomatic Foundation of the Analytic Hierarchy Process.* Management Science **32**, 7 (1986), 841–855.

[8] Saaty, T. L.: *Decision Making with the Analytic Hierarchy Process.* Int. J. Services Sciences **1**, 1 (2008), 83–98.

[9] Saaty, T. L.: *Decision-making with the AHP: Why is the principal eigenvector necessary.* European Journal of Operational Research **145** (2003), 85–91.

[10] Saaty, T. L.: *How to make a decision: The Analytic Hierarchy Process.* European Journal of Operational Research **48** (1990), 9–26.

[11] Saaty, T. L.: *Relative Measurement and Its Generalization in Decision Making, Why Pairwise Comparisons are Central in Mathematics for the Measurement of Intangible Factors – The Analytic Hierarchy/Network Process.* RACSAM **102**, 2 (2008), 251–318.

[12] Stoklasa, J., Jandová, V., Talašová, J.: *Weak consistency in Saaty's AHP – evaluating creative work outcomes of Czech Art Colleges.* Neural Network **23** (2013), 61–77.

[13] Talašová, J.: Fuzzy metody vícekriteriálního hodnocení a rozhodování. *Vydavatelství UP*, Olomouc, 2003.