Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **52**, 1 (2013) 101–120

On Semi-Boolean-Like Algebras

Antonio LEDDA [1a], Francesco PAOLI [1b] Antonino SALIBRA [2]

[1] *Dept. of Pedagogy, Psychology, Philosophy, University of Cagliari, Italy*
[a] *e-mail: antonio.ledda@unica.it*
[b] *e-mail: paoli@unica.it*

[2] *Dept. of Environmental Science, Computer Science and Statistics, Università Ca'Foscari, Venice, Italy*
e-mail: salibra@dsi.unive.it

(Received February 1, 2013)

Abstract

In a previous paper, we introduced the notion of *Boolean-like algebra* as a generalisation of Boolean algebras to an arbitrary similarity type. In a nutshell, a double-pointed algebra $\mathbf{A}$ with constants $0, 1$ is Boolean-like in case for all $a \in A$ the congruences $\theta(a, 0)$ and $\theta(a, 1)$ are complementary factor congruences of $\mathbf{A}$. We also introduced the weaker notion of *semi-Boolean-like algebra*, showing that it retained some of the strong algebraic properties characterising Boolean algebras. In this paper, we continue the investigation of semi-Boolean like algebras. In particular, we show that every idempotent semi-Boolean-like variety is term equivalent to a variety of noncommutative Boolean algebras with additional regular operations.

Key words: Boolean-like algebra, central element, noncommutative lattice theory

2000 Mathematics Subject Classification: 03C05, 06E75

1 Introduction

If asked to mention examples of varieties that are especially well-behaved from the viewpoint of their algebraic properties, most universal algebra practicioners would probably include Boolean algebras in their list. More than that, Boolean algebras would most likely be cited as the leading example of a well-behaved *double-pointed* variety—meaning a variety $\mathcal{V}$ whose type includes two constants $0, 1$ such that $0^{\mathbf{A}} \neq 1^{\mathbf{A}}$ in every nontrivial $\mathbf{A} \in \mathcal{V}$. Yet, since it is not infrequent to find other double-pointed varieties of algebras that display Boolean-like features, it can be reasonably asked what common properties of such are responsible for this kind of behaviour. In [15], we introduced the notion of *Boolean-like algebra* as a generalisation of Boolean algebras to a double-pointed but otherwise arbitrary similarity type. The idea behind our approach was that a Boolean-like algebra is an algebra $\mathbf{A}$ such that every $a \in A$ is *central* in the sense of Vaggione [17], meaning that $\theta(a, 0)$ and $\theta(a, 1)$ are complementary factor congruences of $\mathbf{A}$. Central elements are especially convenient to work with because they can be given an expedient equational characterisation; it also turns out that some important properties of Boolean algebras are shared not only by Boolean-like algebras, but also by algebras whose elements satisfy all the equational conditions of central elements but one—in the terminology of [15], algebras whose elements are all *semi-central*. These algebras, and the varieties they form, were termed *semi-Boolean-like* in the same paper. Although double-pointed discriminator varieties are prime examples of semi-Boolean-like varieties, a generic semi-Boolean-like variety need not be c-subtractive or c-regular ($c \in \{0, 1\}$). A better approximation to double-pointed discriminator varieties is given by *idempotent* semi-Boolean-like varieties, whose members are 0-subtractive; actually, a double-pointed variety is discriminator iff it is idempotent semi-Boolean like and 0-regular [15, Theorem 5.6]. This theorem also yields a new Maltsev-type characterisation of double-pointed discriminator varieties.

In the present paper, we establish some new results on semi-Boolean-like algebras and varieties. In § 2, we recall from [13] the notions of *Church algebra* and *Church variety* as the most general concepts on which our approach is based. Essentially, a Church algebra is a double-pointed algebra that appropriately represents the if-then-else operation by means of a ternary term operation q. In every Church algebra $\mathbf{A}$ central elements are the universe of a Boolean algebra that is isomorphic to the Boolean algebra of complementary factor congruences of $\mathbf{A}$ [15, Theorem 3.7], which means that every central element induces a direct decomposition of $\mathbf{A}$. Here, we prove that the factors in this decomposition can be described by exploiting a generalisation of the relativisation construction for Boolean algebras. In § 3, we proceed to deal with the stronger notions of semi-Boolean-like algebras and varieties. The new results we present are essentially two: we point out the exact relationship between semi-Boolean-like varieties and the quasi-discriminator varieties of [14], and we provide semi-Boolean-like algebras with an explicit weak Boolean product representation with directly indecomposable factors. Finally, idempotent semi-Boolean-like algebras are the centre of § 4. We consider a noncommutative generalisation of Boolean algebras and prove—along the lines of similar results available for pointed discriminator varieties [1] or for varieties with a commutative ternary deduction term [2]—that every idempotent semi-Boolean-like variety is term equivalent to a variety of noncommutative Boolean algebras with additional operations.

To avoid overtly deferring the presentation of the main new ideas and results in this paper, we do not include a Preliminaries section. Still, each individual section is written in such a way as to be reasonably self-contained. In particular, the main definitions and results from [15] are duly recalled and summarised. The notational and terminological conventions in the paper are the standard ones in universal algebra (see e.g. [3]).

2 Church varieties

Although the focus of the present paper is on semi-Boolean-like algebras and varieties, we include herein a theorem on the weaker notion of Church algebra that may yield a better insight into its overall significance. After recalling the relevant definitions and results, in fact, we adapt to Church algebras a variant of the well-known relativisation construction for Boolean algebras (see e.g. [10]).

2.1 Preliminaries

The key observation motivating the introduction of *Church algebras* [13] is that many algebras arising in completely different fields of mathematics—including Heyting algebras, rings with unit, or combinatory algebras—have a term operation q satisfying the fundamental properties of the if-then-else connective: $q(1, x, y) \approx x$ and $q(0, x, y) \approx y$. As simple as they may appear, these properties are enough to yield rather strong results. This motivates the next definition.

Definition 1 An algebra $\mathbf{A}$ of type ν is a *Church algebra* if there are term definable elements $0^{\mathbf{A}}, 1^{\mathbf{A}} \in A$ and a term operation $q^{\mathbf{A}}$ s.t., for all $a, b \in A$

$$q^{\mathbf{A}}\left(1^{\mathbf{A}}, a, b\right) = a \text{ and } q^{\mathbf{A}}\left(0^{\mathbf{A}}, a, b\right) = b.$$

A variety $\mathcal{V}$ of type ν is a *Church variety* if every member of $\mathcal{V}$ is a Church algebra with respect to the same term $q(x, y, z)$ and the same constants 0, 1.

Henceforth, the superscript in $q^{\mathbf{A}}$ will be dropped whenever the difference between the operation and the operation symbol is clear from the context, and a similar policy will be followed in similar cases throughout the paper.

Examples of Church algebras include FL_{ew}-algebras (commutative, integral and double-pointed residuated lattices, for which see [7]) and, in particular, Heyting algebras and thus also Boolean algebras; ortholattices; rings with unit; combinatory algebras. Expanding on an idea due to Vaggione [17], we also define:

Definition 2 An element e of a Church algebra $\mathbf{A}$ is called *central* if the pair $(\theta(e, 0), \theta(e, 1))$ is a pair of complementary factor congruences on $\mathbf{A}$. A central element e is *nontrivial* if $e \notin \{0, 1\}$. By $\mathrm{Ce}(A)$ we denote the *centre* of $\mathbf{A}$, i.e. the set of central elements of the algebra $\mathbf{A}$.

It is proved in [15] that Church algebras have Boolean factor congruences and that, by defining

$$x \wedge y = q(x, y, 0),\ x \vee y = q(x, 1, y) \text{ and } x' = q(x, 0, 1),$$

we get:

Theorem 1 *Let* $\mathbf{A}$ *be a Church algebra. Then* $c[\mathbf{A}] = (\mathrm{Ce}(A); \vee, \wedge, ', 0, 1)$ *is a Boolean algebra which is isomorphic to the Boolean algebra of factor congruences of* $\mathbf{A}$.

It clearly follows that a Church algebra is directly indecomposable iff $\mathrm{Ce}\,(A) = \{0, 1\}$. This result, together with theorems by Comer [6] and Vaggione [17], implies:

Theorem 2 *Let* $\mathbf{A}$ *be a Church algebra,* S *be the Boolean space of maximal ideals of* $c\,[\mathbf{A}]$ *and* $f\colon A \to \Pi_{I\in S} A/\theta_I$ *be the map defined by*

$$f(a) = (a/\theta_I \colon I \in S),$$

where $\theta_I = \bigvee_{e\in I} \theta(0, e)$*. Then we have:*

1. f *gives a weak Boolean representation of* $\mathbf{A}$*.*
2. f *provides a Boolean representation of* $\mathbf{A}$ *iff, for all* $a \neq b \in A$*, there exists a least central element* e *such that* $q(e, a, b) = a$ *(i.e.,* $(a, b) \in \theta(0, e)$*).*

In general, not much can be said about the factors in this representation. However, these factors are guaranteed to be directly indecomposable provided that the d.i. members of $\mathcal{V}$ form a universal class. In fact, following [17], it is shown in [15] that:

Theorem 3 *Let* $\mathcal{V}$ *be a Church variety of type* ν*. Then, the following conditions are equivalent:*

(i) For all $\mathbf{A} \in \mathcal{V}$*, the stalks* $\mathbf{A}/\theta_I$ *(*$I \in S$ *a maximal ideal) are directly indecomposable.*

(ii) The class $\mathcal{V}_{DI}$ *of directly indecomposable members of* $\mathcal{V}$ *is a universal class.*

2.2 A relativisation construction

We begin by recalling the following proposition from [15]:

Proposition 1 *If* $\mathbf{A}$ *is a Church algebra of a given type* ν *and* $e \in A$*, the following conditions are equivalent:*

1. e *is central;*
2. $\theta(e, 0) \wedge \theta(e, 1) = \Delta$*;*
3. *for all* $a, b \in A$*,* $q(e, a, b)$ *is the unique element s.t.* $a\theta(e, 1)q(e, a, b)\theta(e, 0)b$*;*
4. *For all* $a, b \in A$*, for all* n*-ary* $f \in \nu$*, and for all* $\overline{a}, \overline{b} \in A^n$*:*
   1. $q\,(e, a, a) = a$
   2. $q\,(e, q\,(e, a, b)\,, c) = q\,(e, a, c) = q\,(e, a, q\,(e, b, c))$
   3. $q\left(e, f\,(\overline{a})\,, f\,(\overline{b})\right) = f\,(q\,(e, a_1, b_1)\,, \ldots, q\,(e, a_n, b_n))$
   4. $q\,(e, 1, 0) = e$
5. *The function* $f_e\,(a, b) = q\,(e, a, b)$ *is a decomposition operation on* $\mathbf{A}$ *s.t.* $f_e\,(1, 0) = e$*.*

In sum, every central element e in a Church algebra yields a direct decomposition of such via the corresponding factor congruences $\theta(e,1)$ and $\theta(e,0)$. The factors $\mathbf{A}/\theta(e,1)$ and $\mathbf{A}/\theta(e,0)$ in this decomposition can be obtained in a simplified way through a variant of the relativisation construction for Boolean algebras.

If $\mathbf{A}$ is a Church algebra of type ν and $e \in A$ is a central element, then we define $\mathbf{A}_e = (A_e; g_e)_{g\in\nu}$ to be the ν-algebra defined as follows:

$$A_e = \{e \wedge b \colon b \in A\}; \quad g_e(e \wedge \overline{b}) = e \wedge g(e \wedge \overline{b}).$$

Theorem 4 *Let* $\mathbf{A}$ *be a Church algebra of type* ν *and* e *be a central element. Then we have:*

1. *For every* n*-ary* $g \in \nu$ *and every sequence of elements* $\overline{b} \in A^n$, $e \wedge g(\overline{b}) = e \wedge g(e \wedge \overline{b})$*, so that the function* $h \colon A \to A_e$*, defined by* $h(b) = e \wedge b$*, is a homomorphism from* $\mathbf{A}$ *onto* $\mathbf{A}_e$*.*
2. $\mathbf{A}_e$ *is isomorphic to* $\mathbf{A}/\theta(e,1)$*. It follows that* $\mathbf{A} = \mathbf{A}_e \times \mathbf{A}_{e'}$ *for every central element* e*, as in the Boolean case.*

Proof (1)

$$\begin{aligned}
e \wedge g(e \wedge \overline{b}) &= q(e, g(e \wedge b_0, \dots, e \wedge b_{n-1}), 0) && \text{Def. } \wedge \\
&= q(e, g(q(e, b_0, 0), \dots, q(e, b_{n-1}, 0)), 0) && \text{Def. } \wedge \\
&= q(e, q(e, g(\overline{b}), g(\overline{0})), 0) && \text{Pr.1.4(3)} \\
&= q(e, g(\overline{b}), 0) && \text{Pr.1.4(2)} \\
&= e \wedge g(\overline{b}) && \text{Def. } \wedge
\end{aligned}$$

$$\begin{aligned}
h(g(\overline{b})) &= e \wedge g(\overline{b}) && \text{Def. } h \\
&= e \wedge g(e \wedge \overline{b}) && \text{First part of the proof} \\
&= g_e(e \wedge \overline{b}) && \text{Def. } g_e \\
&= g_e(h(\overline{b})) && \text{Def. } h
\end{aligned}$$

(2) By (1) we have to show that the kernel of h is the congruence $\theta(e,1)$. Let $h(b) = h(c)$, i.e. $e \wedge b = e \wedge c$. Recall that, by Proposition 1.3, $q(e,b,c)$ is the unique element u such that $b\theta(e,1)u\theta(e,0)c$. Since $b\theta(e,1)q(e,b,0)\theta(e,0)0$ and $c\theta(e,1)q(e,c,0)\theta(e,0)0$, then by $e \wedge b = e \wedge c$ we obtain the conclusion

$$b \ \theta(e,1) \ e \wedge b = e \wedge c \ \theta(e,1) \ c,$$

i.e., $b\theta(e,1)c$.

In the opposite direction, we must show that $h(e) = e \wedge e = q(e,e,0)$ is equal to $h(1) = e \wedge 1 = q(e,1,0) = e$ (because e is central). However, by Proposition 1.4(2), $q(e,e,0) = q(e, q(e,1,0), 0) = q(e,1,0) = e$. □

3 Semi-Boolean-like varieties

After recalling the notions of semi-Boolean-like algebras and varieties and the main results concerning them, we compare these concepts with another generalisation of discriminator varieties, namely the *quasi-discriminator varieties* investigated in [14]. A weak Boolean product representation of semi-Boolean-like algebras that is more informative than the general result stated above in Theorems 2 and 3 concludes the section.

3.1 Preliminaries

In a generic Church algebra, of course, there is no need for the set of central elements to comprise all elements of the algebra — not any more than an arbitrary ortholattice needs to be a Boolean algebra, or a ring with unit a Boolean ring. In [15], Church algebras where the set of central elements comprises all elements of the algebra were introduced and investigated under the label of *Boolean-like algebras*, while the name of *semi-Boolean-like algebras* was reserved for the concept defined below:

Definition 3 We say that a Church algebra $\mathbf{A}$ of type ν is a *semi-Boolean-like algebra* (or a SBlA, for short) if it satisfies the following equations, for all $e, a, a_1, a_2 \in A$, for every n-ary $g \in \nu$, and for every $\overline{b}, \overline{c} \in A^n$:

Ax_0. $q(1, a, b) = a = q(0, b, a)$

Ax_1. $q(e, a, a) = a$

Ax_2. $q(e, q(e, a_1, a_2), a) = q(e, a_1, a) = q(e, a_1, q(e, a_2, a))$

Ax_3. $q(e, g(\overline{b}), g(\overline{c})) = g(q(e, b_1, c_1), \ldots, q(e, b_n, c_n))$.

If $\mathbf{A}$ satisfies Ax_0–Ax_3 plus

Ax_4 $q(a, 1, 0) = a$

then we say that $\mathbf{A}$ is a *Boolean-like algebra* (or a BlA, for short).

Definition 4 A variety $\mathcal{V}$ of type ν is a *(semi-)Boolean-like variety* if every member of $\mathcal{V}$ is a (semi-)Boolean-like algebra with respect to the same term $q(x, y, z)$ and the same constants 0, 1.

It turns out that, if we define $c(x) = q(x, 1, 0)$, an element a in a SBlA is central just in case $c(a) = a$. By Ax_4, therefore, BlAs are precisely those SBlAs where every element is central. The next lemmas and proposition from [15] will be useful in what follows:

Lemma 1 *Let* $\mathbf{A}$ *be a* SBlA. *Then for all* $a, b, d \in A$, $q(a', b, d) = q(a, d, b)$.

Lemma 2 *Let* $\mathbf{A}$ *be a directly indecomposable* SBlA. *Then the following conditions hold for all* $a, b, d \in A$*:*

$$q(a, b, d) = \begin{cases} d & \text{if } c(a) = 0 \\ b & \text{if } c(a) = 1 \end{cases} \quad a' = q(a, 0, 1) = \begin{cases} 1 & \text{if } c(a) = 0 \\ 0 & \text{if } c(a) = 1 \end{cases}$$

$$a \vee b = \begin{cases} b & \text{if } c(a) = 0 \\ 1 & \text{if } c(a) = 1 \end{cases}$$

Proposition 2 *For a Church variety $\mathcal{V}$ (w.r.t. the term q), the following are equivalent:*

1. *$\mathcal{V}$ is semi-Boolean like;*
2. *$\mathcal{V}$ satisfies the conditions:*
   (i) for all $a, b, c \in \mathbf{A} \in \mathcal{V}$, $q(a, b, c) = q(c(a), b, c)$;
   (ii) for all $a \in \mathbf{A} \in \mathcal{V}$, $c(a)$ is central;
3. *$\mathcal{V}$ satisfies the condition 2(i) and the universal formula*

$$c(0) \approx 0 \ \barwedge \ c(1) \approx 1 \ \barwedge \ \forall x (c(x) \approx 0 \veebar c(x) \approx 1)$$

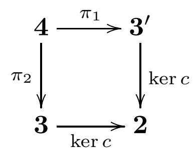
   holds in every s.i. member of $\mathcal{V}$.

The "pure semi-Boolean-like" variety $\mathcal{SB}l\mathcal{A}_0$, consisting of all the term reducts of the form $(A; q, 0, 1)$ of SBlAs, and axiomatised by Ax_0-Ax_3 above, is of independent interest. We say that a term t is *$\mathcal{V}$-idempotent* if $\mathcal{V} \vDash t(t(x)) \approx t(x)$, and *$\mathcal{V}$-compatible* in case $t^{\mathbf{A}}$ is an endomorphism in every $\mathbf{A} \in \mathcal{V}$. It can be shown that the term c is $\mathcal{SB}l\mathcal{A}_0$-compatible and $\mathcal{SB}l\mathcal{A}_0$-idempotent and thus, if $\mathbf{A}$ is a member of $\mathcal{SB}l\mathcal{A}_0$, $c[\mathbf{A}]$ is a *retract* of $\mathbf{A}$. Examples of members of $\mathcal{SB}l\mathcal{A}_0$ are:

Example 1 Let $\mathbf{3} = (\{0, 1, 2\}; q, 0, 1)$ be the Church algebra completely specified by the stipulation that $q(0, a, b) = q(2, a, b)$ for all $a, b \in \{0, 1, 2\}$. It can be checked that $\mathbf{3}$ is semi-Boolean-like. However, $c(2) = q(2, 1, 0) = 0 \neq 2$. Moreover, $\mathbf{3}$ is a nonsimple subdirectly irreducible algebra, with the middle congruence corresponding to the partition $\{\{1\}, \{0, 2\}\}$. Therefore $V(\mathbf{3})$ is not a discriminator variety, although it is a binary 1-discriminator variety in the sense of [5] with binary 1-discriminator term $y' \vee x$.

Example 2 Let $\mathbf{3}' = (\{0, 1, 2\}; q, 0, 1)$ be the Church algebra completely specified by the stipulation that $q(1, a, b) = q(2, a, b)$ for all $a, b \in \{0, 1, 2\}$. It can be checked that $\mathbf{3}'$ is semi-Boolean-like. However, $c(2) = q(2, 1, 0) = 1 \neq 2$. Moreover, $\mathbf{3}'$ is a nonsimple subdirectly irreducible algebra, with the middle congruence corresponding to the partition $\{\{0\}, \{1, 2\}\}$. Therefore $V(\mathbf{3}')$ is not a discriminator variety, although it is a binary 0-discriminator variety with binary 0-discriminator term $y' \wedge x$.

The algebras we just introduced are actually more than workaday examples of pure SBlAs. In fact, let $\mathbf{4}$ be the fibred product $\mathbf{3} \times_{\mathbf{2}} \mathbf{3}'$, i.e. the algebra whose universe is $\{(0, 0), (2, 0), (1, 2), (1, 1)\}$ and whose factorisation is described by the following self-explanatory diagram:



We have that:

Theorem 5 $\mathcal{SB}l\mathcal{A}_0 = V(\{\mathbf{3}, \mathbf{3}'\}) = V(\mathbf{4})$.

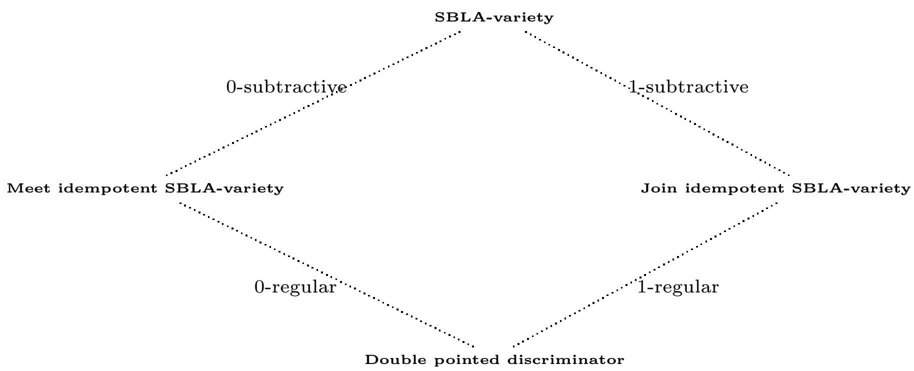
$\mathcal{SB}l\mathcal{A}_0$ has three proper nontrivial subvarieties:

- $\mathcal{ISB}l\mathcal{A}_0$, the subvariety generated by $\mathbf{3}'$, whose equational basis relative to $\mathcal{SB}l\mathcal{A}_0$ is given by the single identity $x \wedge x \approx x$;
- $\mathcal{JSB}l\mathcal{A}_0$, the subvariety generated by $\mathbf{3}$, whose equational basis relative to $\mathcal{SB}l\mathcal{A}_0$ is given by the single identity $x \vee x \approx x$;
- $\mathcal{B}l\mathcal{A}_0$, the variety consisting of all the term reducts of the form $(A; q, 0, 1)$ of BlAs, generated by the two element BlA, whose equational basis relative to $\mathcal{SB}l\mathcal{A}_0$ is given either by the single identity $x \wedge y \approx y \wedge x$, or by the two idempotency identities $x \wedge x \approx x, x \vee x \approx x$, or by the identity $c(x) \approx x$.

More generally, a semi-Boolean-like variety is said to be *meet idempotent* if it satisfies the identity

$$Ax_5 \colon x \wedge x \approx x.$$

(It is said to be *join idempotent* if it satisfies the identity $x \vee x \approx x$). Although an arbitrary semi-Boolean-like variety needs not be c-subtractive or c-regular ($c \in \{0, 1\}$), a meet (join) idempotent semi-Boolean-like variety is always 0- (1-)subtractive with witness term $y' \wedge x$ ($y' \vee x$). Adding 0- (1-)regularity constraints to the preceding concepts suffices to deliver double pointed discriminator varieties, according to the following schema:



To be slightly more specific than the previous diagram allows us to be, the next theorem characterises meet idempotent semi-Boolean-like varieties in the context of semi-Boolean-like varieties:

Theorem 6 *Let $\mathcal{V}$ be a semi-Boolean-like variety. Then the following conditions are equivalent:*

(i) $\mathcal{V}$ is meet idempotent;

(ii) $\mathcal{V}$ *is a unary discriminator variety*[*] *w.r.t.* c*;*

(iii) The identity $x \vee x \approx c(x)$ *holds in* $\mathcal{V}$*;*

(iv) $\mathcal{V}$ *is* 0*-subtractive with witness term* $y' \wedge x$*.*

Henceforth, consistently with our introduction, we will use the abbreviation *idempotent* in place of the more cumbersome *meet idempotent*.

3.2 Quasi-discriminator varieties

To some extent, as mentioned above, semi-Boolean-like varieties generalise discriminator varieties in the double-pointed case. A different generalisation of discriminator varieties was suggested in [14] with the aim of explaining why some varieties of algebras, which fail to be discriminator varieties, retain some of the pleasing properties of such varieties nonetheless. The following concept, currently under investigation, provides a significant clue towards a satisfactory answer to this question.

Definition 5 Let $\mathcal{V}$ be a variety whose type ν includes a unary term $\Box$. Moreover, suppose that $\Box$ is $\mathcal{V}$-idempotent and $\mathcal{V}$-compatible. $\mathcal{V}$ is a *quasi-discriminator variety* w.r.t. $\Box$ if there is a quaternary term s of type ν s.t., for every s.i. member $\mathbf{A}$ of $\mathcal{V}$ and for all $a, b, c, d \in A$,

$$s\,(a, b, c, d) = \begin{cases} c \text{ if } \Box a = \Box b \\ d \text{ if } \Box a \neq \Box b \end{cases}$$

Of course, discriminator varieties are a special case of quasi-discriminator varieties when $\Box$ is the identity. A variety which is quasi-discriminator w.r.t. a nonidentity unary term is called *properly* quasi-discriminator. Examples of properly quasi-discriminator varieties include Gödel algebras, product algebras and other varieties of fuzzy logic [8], as well as regular Nelson residuated lattices [4]. We now prove that:

Theorem 7 *For a double pointed variety* $\mathcal{V}$*, the following are equivalent:*

1. *(i)* $\mathcal{V}$ *is semi-Boolean-like w.r.t. the ternary term* q *and*

   (ii) c *is* $\mathcal{V}$*-compatible;*

2. *(i)* $\mathcal{V}$ *is a quasi-discriminator variety w.r.t. the unary term* c*, satisfying* $c(0) \approx 0$ *and* $c(1) \approx 1$*, and*

   (ii) the universal formula

$$\forall x (c(x) \approx 0 \veebar c(x) \approx 1) \tag{3.1}$$

   holds in every directly indecomposable member of $\mathcal{V}$*.*

[*] Recall that the unary discriminator on a double pointed set A (with constants $0, 1$) is a unary function u on A such that $u(0) = 0$ and $u(a) = 1$ for $a \neq 0$. A variety $V(\mathcal{K})$ of type ν is a unary discriminator variety iff there is a unary term of type ν realising the unary discriminator in all members of $\mathcal{K}$.

Proof (1) implies (2). Let $x \leftrightarrow_c y$ be defined as $(x' \vee c(y)) \wedge (y' \vee c(x))$, and set

$$s(x, y, z, w) = q(x \leftrightarrow_c y, z, w)$$

Observe that in a directly indecomposable algebra $c(x) = c(y)$ iff $c(x \leftrightarrow_c y) = c(x) \leftrightarrow_c c(y) = 1$. Then, using Lemma 2 and Proposition 2, we get our conclusion.

(2) implies (1). Let $q(x, y, z) = s(x, 0, z, y)$. We observe the following facts:

(a) $\mathcal{V}$ satisfies $c(0) \approx 0$ and $c(1) \approx 1$ by our hypothesis;

(b) $\mathcal{V}$ is a Church variety, because $q(0, a, b) = s(0, 0, b, a) = b$ and $q(1, a, b) = s(1, 0, b, a) = a$;

(c) $\mathcal{V} \vDash s(x, 0, z, y) \approx s(c(x), 0, z, y)$ as this identity holds in all s.i. members of $\mathcal{V}$, if we take into account the idempotency of c.

(d) $q(x, y, z) \approx q(c(x), y, z)$ holds by (c) and by the definition of q.

(e) $\mathcal{V} \vDash c(x) \approx q(x, 1, 0)$ (whence the term c given by our hypothesis coincides with the term c as usually defined). In fact, let $a \in A$, where $\mathbf{A}$ is a s.i. member of $\mathcal{V}$. Then

$$q(a, 1, 0) = s(a, 0, 0, 1) = s(c(a), 0, 0, 1) = \begin{cases} 0 \text{ if } c(a) = 0 \\ 1 \text{ if } c(a) = 1 \end{cases}.$$

(f) By (e), $\mathcal{V} \vDash q(x, y, z) \approx q(c(x), y, z)$.

Our conclusion now follows from Proposition 2. □

Observe that, if $\mathcal{V}$ has type $(3, 0, 0)$, then Theorem 7.(2) above is equivalent to Theorem 7.(1)(i) alone, because in this case c is $\mathcal{V}$-compatible whenever $\mathcal{V}$ is semi-Boolean like.

3.3 A representation theorem

Theorems 2 and 3 imply that every SBlA has a weak Boolean product representation with directly indecomposable factors, because the directly indecomposable members of any SBlA variety can be defined via the universal formula

$$\forall x \, (x \approx c(x) \Rightarrow x \approx 0 \veebar x \approx 1).$$

A more informative proof of this fact, however, can be given by modifying a construction from [9] to the case that is of interest to us. Thus, let $\mathbf{A}$ be a SBlA of type ν, and I a Boolean ideal of the Boolean algebra $c[\mathbf{A}]$ of central elements of $\mathbf{A}$. We start with a useful lemma.

Lemma 3 *Let $f, g \in \mathrm{Ce}(A)$, $f \leq g$, and let $q(g, a, b) = b$. Then $q(f, a, b) = b$.*

Proof Observe that if $f \leq g$ in $c[\mathbf{A}]$, then $q(g, f, 0) = f$, for $\wedge^{c[\mathbf{A}]}$ coincides with lattice meet. So

$$\begin{aligned} q(f, a, b) &= q(q(g, f, 0), a, b) \\ &= q(q(g, f, 0), q(g, a, a), q(g, a, b))) && \text{Hp., Ax}_1 \\ &= q(g, q(f, a, a), q(0, a, b)) && \text{Ax}_3 \\ &= q(g, a, b) && \text{Ax}_0\text{, Ax}_1 \\ &= b && \text{Hp.} \end{aligned}$$

□

We now define

$$\theta_I = \{(a,b) \in A^2 \colon \exists e\, (e \in I \ \barwedge q(e,a,b) = a)\}.$$

Lemma 4 *The relation θ_I is a congruence.*

Proof By Proposition 1

$$\theta_I = \{(a,b) \in A^2 \colon \exists e\, (e \in I \ \barwedge (a,b) \in \theta(e,0))\} = \bigvee_{e \in I} \theta(0,e).$$

Now, $\bigvee_{e\in I} \theta(0,e)$ is a congruence for I is an ideal of $c\,[\mathbf{A}]$. □

Lemma 5 *If $a \neq b \in A$, then there is a maximal ideal I^* on $c[\mathbf{A}]$ s.t. $(a,b) \notin \theta_{I^*}$.*

Proof If $a \neq b$, consider the set $I = \{e \in \mathrm{Ce}(A) \colon q(e,a,b) = b\}$. Clearly, $0 \in I$. Suppose that $e, f \in I$. Then:

$$\begin{aligned} q(e \vee f, a, b) &= q(q(e,1,f),a,b) && \text{Def.} \\ &= q(q(e,1,f), q(e,a,a), q(e,b,b)) && \mathrm{Ax}_1 \\ &= q(e, q(1,a,b), q(f,a,b)) && \mathrm{Ax}_3 \\ &= q(e,a,q(f,a,b)) && \mathrm{Ax}_1 \\ &= q(e,a,b) = b. && \text{Hp.} \end{aligned}$$

If $f \in I$ and $e \leq f$, then by Lemma 3 we obtain that $q(f,a,b) = b$ and thus $e \in I$. Consequently, I is a non-void Boolean ideal of $c\,[\mathbf{A}]$. Therefore, I can be extended to a maximal ideal I^* on $c\,[\mathbf{A}]$. Now, suppose $(a,b) \in \theta_{I^*}$. Then, there is an $e \in I^*$ s.t. $q(e,a,b) = a$ and so $q\,(e,b,a) = b$, whence $e' \in I^*$, which is impossible. Then $(a,b) \notin \theta_{I^*}$. □

Theorem 8 *Every* SBlA $\mathbf{A}$ *is a weak Boolean product of directly indecomposable semi-Boolean-like algebras.*

Proof (i) $\mathbf{A}$ *is a subdirect product of semi-Boolean-like algebras.* Let $\mathcal{I}$ be the collection of all maximal Boolean ideals of $c[\mathbf{A}]$. Let $a, b \in A$, and suppose $(a,b) \in \bigcap \{\theta_I \colon I \in \mathcal{I}\}$. If $a \neq b$, then, by Lemma 5, there is a maximal ideal I^* s.t. $(a,b) \notin \theta_{I^*}$, a contradiction. Therefore, $\bigcap \{\theta_I \colon I \in \mathcal{I}\} = \Delta$, whence $\mathbf{A}$ is subdirectly embeddable into $\prod \{\mathbf{A}/\theta_I \colon I \in \mathcal{I}\}$.

(ii) *The stalks are directly indecomposable.* Assume, by way of contradiction, that $\mathbf{A}/\theta_I$ is directly decomposable. Then there exists a nontrivial central element $a/\theta_I \in \mathbf{A}/\theta_I$. Consider the finite set Π of identities defining central elements in the type of $\mathbf{A}$. For every $t \approx s \in \Pi$, we have in $\mathbf{A}/\theta_I$ that $t\,(a/\theta_I) = s\,(a/\theta_I)$, whence there exists a central element $e \in I$ s.t. $(t(a), s(a)) \in \theta(e,0)$. Define $\varphi = \theta(e,0)$, $\overline{\varphi} = \theta(e,1)$, $\psi = \theta\,(a/\varphi, 0/\varphi)$, $\overline{\psi} = \theta\,(a/\varphi, 1/\varphi)$. Then a/φ is central in $\mathbf{A}/\varphi$. Since $\mathbf{A} = \mathbf{A}/\varphi \times \mathbf{A}/\overline{\varphi}$ and $\mathbf{A}/\varphi = (\mathbf{A}/\varphi)\,/\psi \times (\mathbf{A}/\varphi)\,/\overline{\psi}$, we get the following decomposition of $\mathbf{A}$:

$$\mathbf{A} = (\mathbf{A}/\varphi)\,/\psi \times \left((\mathbf{A}/\varphi)\,/\overline{\psi} \times \mathbf{A}/\overline{\varphi}\right).$$

Therefore, there exists a central element $d \in A$ such that $(\mathbf{A}/\varphi)/\psi = \mathbf{A}/\theta(d,0)$. Now, $(a,0) \in \theta(d,0)$, and, given the way ψ is defined, $a/\varphi = d/\varphi$. We claim that $d \notin I$, for suppose otherwise. Then $\theta(d,0) \subseteq \theta_I$, and we would get a homomorphism from $\mathbf{A}/\theta(d,0)$ to $\mathbf{A}/\theta_I$, which contradicts the fact that a/θ_I is a nontrivial central element, thus establishing $d \notin I$. Since I is maximal, $d' \in I$. So $0/\theta_I = d'/\theta_I$ is the complement of a/θ_I, which implies $a/\theta_I = 1/\theta_I$, a contradiction.

(iii) *The subdirect representation is a weak Boolean representation.* The set $\{I \in \mathcal{I} \colon a_I = b_I\}$ is open because $a_I = b_I$ iff $a\theta_I b$ iff there exists $e \in I$ s.t. $q(e,a,b) = a$ iff $a\theta_J b$ for all J s.t. $e \in J$. Moreover, if U is clopen then there exists $e \in \mathrm{Ce}(A)$ s.t. $U = \{I \in \mathcal{I} \colon e \in I\}$ and, consequently, $\mathcal{I} - U = \{I \in \mathcal{I} \colon e' \in I\}$, but for every such e and given $a, b \in A$, there exists a unique $c \in A$ (namely, $q(e,a,b)$) s.t. $(a,c) \in \theta(e',0)$ and $(b,c) \in \theta(e,0)$. □

4 Noncommutative Boolean Algebras

The significance of pointed discriminator varieties, as well as the extent to which they generalise Boolean algebras in the pointed case, are more perspicuously appreciated in light of the term equivalence result with left-handed skew Boolean $\cap$-algebras proved by Bignall and Leech (Theorem 9 below). Noncommutative Boolean algebras, defined hereafter, play a similar role w.r.t. idempotent semi-Boolean-like varieties, in consideration of the term equivalence result proved in the sequel (Theorem 10). The investigation of noncommutative Boolean algebras is preceded by a brief survey on skew Boolean $\cap$-algebras and other noncommutative generalisations of lattices, included for the reader's convenience.

4.1 Noncommutative lattice theory

Weakenings of lattices where the meet and join operations may fail to be commutative attracted from time to time the attention of various communities of scholars, including ordered algebra theorists (for their connection with preordered sets) and semigroup theorists (who viewed them as structurally enriched bands possessing a dual multiplication). Here we will review some basic definitions and results on one such generalisation, probably the most interesting and successful: the concept of *skew lattice* [11], in fact, along with the related notion of *skew Boolean algebra*, has important connections with discriminator varieties; the interested reader is referred to [12] or [16] for far more comprehensive accounts.

Definition 6 A *band* is a semigroup $(A; \cdot)$ satisfying the identity $xx \approx x$. A band is *regular* if it satisfies $xyxzx \approx xyzx$; it is *left (right) regular* if it satisfies the identity $xyx \approx xy$ ($xyx \approx yx$).

Left and right regular bands are obviously regular. Observe that, given a band $\mathbf{A}$, the relation

$$a \leq b \Leftrightarrow ab = a = ba$$

is a partial ordering on A.

Definition 7 A *double band* is an algebra $(A;+,\cdot)$ of type $(2,2)$ such that the reducts $(A;\cdot)$ and $(A;+)$ are both bands. A double band satisfying the absorption identities

$$x(x+y) \approx x \approx x+xy;$$
$$(y+x)x \approx x \approx yx+x.$$

is called a *skew lattice*. A skew lattice is called *left handed (right handed)* if the reduct $(A;\cdot)$ is left (right) regular and the reduct $(A;+)$ is right (left) regular.

If we expand skew lattices by a subtraction operation and a constant 0, we get the following noncommutative variant of Boolean algebras.

Definition 8 A *skew Boolean algebra* is an algebra $\mathbf{A} = (A;+,\cdot,-,0)$ of type $(2,2,2,0)$ such that:

- its reduct $(A;+,\cdot)$ is a skew lattice satisfying the identities $xyzx \approx xzyx$, $x(y+z) \approx xy+xz$ and $(y+z)x \approx yx+zx$;
- 0 is left and right absorbing w.r.t. multiplication;
- the operation $-$ satisfies the identities

$$xyx+(x-y) \approx x \approx (x-y)+xyx;$$
$$xyx(x-y) \approx 0 \approx (x-y)xyx.$$

Skew Boolean algebras s.t. every finite set of their universe has an infimum w.r.t. the underlying natural partial ordering of the algebra stand out for their significance. It turns out that such algebras can be given an equational characterisation provided we include the binary inf into the signature.

Definition 9 A *skew Boolean $\cap$-algebra* is an algebra $\mathbf{A} = (A;+,\cdot,\cap,-,0)$ of type $(2,2,2,2,0)$ s.t.:

- the reduct $(A;+,\cdot,-,0)$ is a skew Boolean algebra and the reduct $(A;\cap)$ is a meet semilattice;
- $\mathbf{A}$ satisfies the identities

$$x\cap(xyx) \approx xyx;$$
$$x\,(x\cap y) \approx x\cap y \approx (x\cap y)\,x.$$

The next theorem by Bignall and Leech [1] provides a powerful bridge between the theories of skew Boolean algebras and pointed discriminator varieties:

Theorem 9 *(i) The variety of type* $(3,0)$ *generated by the class of all pointed discriminator algebras* $\mathbf{A} = (A;t,0)$*, where* t *is the discriminator function on* A *and* 0 *is a constant, is term equivalent to the variety of left handed skew Boolean* $\cap$*-algebras.*

(ii) Every discriminator variety is term equivalent to a variety $\mathcal{V}$ *of left handed skew Boolean* $\cap$*-algebras with additional operations* $(g_i)_{i\in I}$ *such that, for all algebras* $\mathbf{A}\in\mathcal{V}$*, every congruence* θ *on the term reduct* $(A;t,0)$ *is compatible with* g_i *for all* $i\in I$*.*

4.2 Definition, examples and elementary properties

Definition 10 A *noncommutative Boolean algebra* is an algebra

$$\mathbf{A} = (A; +, \cdot, ', 0, 1)$$

of type $(2, 2, 1, 0, 0)$ satisfying the following conditions for all $a, b, c \in A$:

(S0) $(A; +, \cdot)$ is a double band;

(S1) $0 + a = a = a + 0; \qquad 1 + a = 1;$

(S2) $0a = a0 = 0; \qquad 1a = a;$

(S3) $a(b + c) = ab + ac; \qquad (b + c)a = ba + ca;$

(S4) $(a + b)' = a'b'; \qquad (ab)' = a' + b';$

(S5) $aa' = a'a = 0;$

(S6) $a + a' = a' + a = a + 1;$

(S7) $ba + b'a = a.$

Observe that S1 and S2 respectively say that $(A; +)$ is a unital groupoid with left absorbing element, and that $(A; \cdot)$ is a groupoid with absorbing element and left unit. "Noncommutative Boolean algebra" will be sometimes abbreviated by NBA, and the variety of noncommutative Boolean algebras will be denoted by $\mathcal{NBA}$.

Definition 11 An algebra $\mathbf{A} = (A; +, \cdot, ', 0, 1, g)_{g \in \nu}$ of type ν is said to be a *noncommutative Boolean algebra with additional regular operations* iff it is an NBA satisfying the identities

(S8) $g(\dots, ex_i + e'y_i, \dots) \approx e \cdot g(\dots, x_i, \dots) + e' \cdot g(\dots, y_i, \dots) \ (g \in \nu).$

Lemma 6 *S8 implies the identity* $(xy + x'z)' \approx xy' + x'z'$.

Proof By applying S8 to $'$. □

Example 3 Let $x + y = q(x, x, y)$. The $(+, \wedge, ', 0, 1)$-term reduct of the pure SBlA **4**, i.e. the algebra whose operations are specified by the following tables:

$'$	**0**	**2**	**3**	**1**
	1	0	0	0

$+$	**0**	**2**	**3**	**1**
0	0	2	3	1
2	2	2	2	2
3	3	3	3	3
1	1	1	1	1

$\wedge$	**0**	**2**	**3**	**1**
0	0	0	0	0
2	0	2	3	1
3	0	2	3	1
1	0	2	3	1

is a NBA.

Observe that its subalgebra over the universe $\{0, 2, 1\}$ is the $(+, \wedge, ', 0, 1)$-term reduct of the pure idempotent SBlA $\mathbf{3}'$.

The following lemma contains some useful properties of NBAs.

Lemma 7 *Every* NBA *satisfies the following identities:*

(i) $0' = 1$ *and* $1' = 0$.

(ii) $x + yx \approx x \approx (x + y)x$;

(iii) $yx + x \approx x$;

(iv) $x'' \approx x1$;

(v) $x + xy \approx x \approx x(x + y)$;

(vi) $x + y \approx x + x'y$;

(vii) $xy + x \approx x(y + x)$;

(viii) $xy + xz + y + z \approx y + z$;

(ix) $x(x + y)z \approx x(y + x)z \approx xz$;

(x) $x + y + x \approx x + y$;

(xi) $xyx \approx yx$;

(xii) $x'y + xz + y \approx xz + y$;

(xiii) $xyz \approx yxz$;

(xiv) $x'z + xy \approx xy + x'z$.

Proof For the remainder of the proof, fix a generic NBA **A** and arbitrary elements $a, b, c, d \in A$.

(i) $1' =_{(S2)} 11' =_{(S5)} 0$ and $0' =_{(S1)} 0 + 0' =_{(S6)} 0 + 1 =_{(S1)} 1$.

(ii) $(a + b)\, a =_{(S3)} aa + ba =_{(S0)} a + ba =_{(S2)} 1a + ba =_{(S3)} (1 + b)\, a =_{(S1)} 1a =_{(S2)} a$.

(iii) Similar.

(iv) $a1 =_{(i)} a0' =_{(S5)} a(aa')' =_{(S4)} a(a' + a'') =_{(S3)} (aa') + (aa'') =_{(S5)} 0 + (aa'') =_{(S1)} aa'' =_{(S1)} aa'' + 0 =_{(S5)} aa'' + a'a'' =_{(S7)} a''$.

(v) For a start, $a(a + 1) =_{(S6)} a(a + a') =_{(S3)} aa + aa' =_{(S0)} a + aa' =_{(S5)} a + 0 =_{(S1)} a$. Therefore, $a + ab = a(a + 1) + ab =_{(S3)} a((a + 1) + b) =_{(S0)} a(a + (1 + b)) =_{(S1)} a(a + 1) = a$. The remaining statement is proved as in (ii).

(vi) $a + b =_{(S7)} a + (ab + a'b) =_{(S0)} (a + ab) + a'b =_{(v)} a + a'b$.

(vii) By (S3) and (S0).

(viii) $ab + ac + b + c =_{(S3)} a\,(b + c) + b + c =_{(iii)} b + c$.

(ix) $a(a + b)c = ac$ by (v). $a(b + a)c =_{(vii)} (ab + a)c =_{(S3)} abc + ac =_{(S3)} a(bc + c) =_{(iii)} ac$.

(x) $a + b + a =_{(v)} a + b + a(a + b) =_{(ii)} a + b$.

(xi) $aba =_{(ii)} (a + ba)ba =_{(S0,S3)} aba + ba =_{(iii)} ba$.

(xii) $a'b + ac + b =_{(S1,S2,S5)} a'ac + a'b + ac + b =_{(viii)} ac + b$.

(xiii) $bac =_{(xi)} abac =_{(ix)} aba(b + a)c =_{(vii)} ab(ab + a)c =_{(ix)} abc$.

(xiv) $a'c + ab =_{(xii)} a''ab + a'c + ab =_{(iv)} a1ab + a'c + ab =_{(S0,S2)} ab + a'c + ab =_{(x)} ab + a'c$. □

By Lemma 7.(x),(xi), the reducts $(A; \cdot)$ and $(A; +)$ in an NBA are, respectively, right regular and left regular as bands.

4.3 A term equivalence result

Our next task on the agenda is trying to obtain an analogue of Theorem 9 for idempotent semi-Boolean-like varieties. More precisely, we will prove that every such variety is term equivalent to a variety of NBAs with additional regular operations.

Consider the following correspondences between the algebraic similarity types of $\mathcal{NBA}$ and of $\mathcal{SB}l\mathcal{A}_0$:

$$\begin{array}{lcl} q(x,y,z) & \rightsquigarrow & xy + x'z \\ x+y & \rightsquigarrow & q(x,x,y) \\ xy & \rightsquigarrow & q(x,y,0) \\ x' & \rightsquigarrow & q(x,0,1) \end{array}$$

Theorem 10 *The above correspondences define a term equivalence between the varieties $\mathcal{ISB}l\mathcal{A}_0$ and $\mathcal{NBA}$.*

Proof If $\mathbf{A}$ is a pure idempotent SBlA and $\mathbf{B}$ is an NBA, we define

$$\mathbf{A}^a = (A; +, \wedge, ', 0, 1), \qquad \mathbf{B}^- = (B; q, 0, 1),$$

where $q(x,y,z) = xy + x'z$. We have to prove that:

(i) if $\mathbf{A}$ is a member of $\mathcal{ISB}l\mathcal{A}_0$, then $\mathbf{A}^a$ is an NBA;

(ii) if $\mathbf{B}$ is an NBA, then $\mathbf{B}^-$ is a member of $\mathcal{ISB}l\mathcal{A}_0$;

(iii) $(\mathbf{A}^a)^- = \mathbf{A}$ and $(\mathbf{B}^-)^a = \mathbf{B}$.

(i) By way of example, let us check (S7).

$$\begin{array}{rcll} ba + b'a & = & q\left(q(b,a,0), q(b,a,0), q\left(b',a,0\right)\right) & \text{Def.} \\ & = & q\left(q(b,a,0), q(b,a,0), q(b,0,a)\right) & \text{L.1} \\ & = & q\left(b, q\left(a,a,0\right), q\left(0,0,a\right)\right) & \text{Ax}_3 \\ & = & q(b,a,a) & \text{Ax}_5,\text{Ax}_0 \\ & = & a & \text{Ax}_1 \end{array}$$

(ii) Let $\mathbf{B} = (B; +, \cdot, ', 0, 1)$ be an NBA. We prove the axioms of $\mathcal{ISB}l\mathcal{A}_0$ for the algebra $\mathbf{B}^-$. Throughout this proof, let $a, b, \dots$ be generic elements of B.

(Ax_0)

$$\begin{array}{rcll} q(1,a,b) & = & 1a + 1'b & \text{Def.} \\ & = & 1a + 0b & \text{L.7} \\ & = & a + 0 = a & \text{S2, S1} \end{array}$$

$$\begin{array}{rcll} q(0,a,b) & = & 0a + 0'b & \text{Def.} \\ & = & 0 + 1b & \text{S2, L.7} \\ & = & 0 + b = b & \text{S2} \end{array}$$

(Ax_1) By S7, $q(a,b,b) = ab + a'b = b$.

(Ax_2)

$$\begin{aligned} q(a,q(a,b,c),d) &= a(ab+a'c)+a'd && \text{Def.}\\ &= aab+aa'c+a'd && \text{S3}\\ &= ab+0c+a'd && \text{S0, S5}\\ &= ab+a'd && \text{S1, S2}\\ &= q(a,b,d) \end{aligned}$$

$$\begin{aligned} q(a,b,q(a,c,d)) &= ab+a'(ac+a'd) && \text{Def.}\\ &= ab+a'ac+a'a'd && \text{S3}\\ &= ab+a'd && \text{S0, S1, S5}\\ &= q(a,b,d) \end{aligned}$$

(Ax_5) $a \wedge a = q(a,a,0) = aa + a'0 = aa = a$ by S0, S1 and S2.

(Ax_3) can be taken w.l.g. to denote an equality of the following form for the pure variety:

$$q(a, b_1c_1 + b_1'd_1, b_2c_2 + b_2'd_2) = q(ab_1 + a'b_2, ac_1 + a'c_2, ad_1 + a'd_2).$$

In other words, we have to show that:

$$a(b_1c_1+b_1'd_1)+a'(b_2c_2+b_2'd_2) = (ab_1+a'b_2)(ac_1+a'c_2)+(ab_1+a'b_2)'(ad_1+a'd_2)$$

Let r and s be the elements denoted, respectively, by the right-hand side and by the left-hand side of the previous equality. Then:

$$\begin{aligned} r &= (ab_1+a'b_2)(ac_1+a'c_2)+(ab_1+a'b_2)'(ad_1+a'd_2)\\ &= (ab_1+a'b_2)(ac_1+a'c_2)+(ab_1)'(a'b_2)'(ad_1+a'd_2) && \text{S4}\\ &= (ab_1+a'b_2)(ac_1+a'c_2)+(a'+b_1')(a''+b_2')(ad_1+a'd_2) && \text{S4}\\ &= (ab_1+a'b_2)ac_1+(ab_1+a'b_2)a'c_2\\ &\quad +(a'+b_1')(a''+b_2')(ad_1+a'd_2) && \text{S3}\\ &= ab_1ac_1+a'b_2ac_1+ab_1a'c_2+a'b_2a'c_2\\ &\quad +(a'+b_1')(a''+b_2')(ad_1+a'd_2) && \text{S3}\\ &= ab_1ac_1+a'ab_2c_1+aa'b_1c_2+a'b_2a'c_2+(a'+b_1')(a''\\ &\quad +b_2')(ad_1+a'd_2) && \text{L.7(xiii)}\\ &= ab_1ac_1+0+0+a'b_2a'c_2+(a'+b_1')(a''+b_2')(ad_1+a'd_2) && \text{S2,S5}\\ &= ab_1ac_1+a'b_2a'c_2+(a'+b_1')(a''+b_2')(ad_1+a'd_2) && \text{S1}\\ &= ab_1c_1+a'b_2c_2+(a'+b_1')(a''+b_2')(ad_1+a'd_2) && \text{L.7(xiii),S0}\\ &= ab_1c_1+a'b_2c_2+((a'+b_1')a''+(a'+b_1')b_2')(ad_1+a'd_2) && \text{S3}\\ &= ab_1c_1+a'b_2c_2+(a'a''+b_1'a''+a'b_2'+b_1'b_2')(ad_1+a'd_2) && \text{S3}\\ &= ab_1c_1+a'b_2c_2+(b_1'a''+a'b_2'+b_1'b_2')(ad_1+a'd_2) && \text{S1,S5}\\ &= ab_1c_1+a'b_2c_2+b_1'a''ad_1+a'b_2'ad_1+b_1'b_2'ad_1+b_1'a''a'd_2\\ &\quad +a'b_2'a'd_2+b_1'b_2'a'd_2 && \text{S3}\\ &= ab_1c_1+a'b_2c_2+b_1'ad_1+a'b_2'ad_1+b_1'b_2'ad_1+b_1'a''a'd_2\\ &\quad +a'b_2'a'd_2+b_1'b_2'a'd_2 && \text{L.7(iv), S1, S0}\\ &= ab_1c_1+a'b_2c_2+b_1'ad_1+0+b_1'b_2'ad_1+0+a'b_2'a'd_2\\ &\quad +b_1'b_2'a'd_2 && \text{S5,L.7(xiii), S2}\\ &= ab_1c_1+a'b_2c_2+b_1'ad_1+b_1'b_2'ad_1+a'b_2'a'd_2+b_1'b_2'a'd_2 && \text{S1}\\ &= ab_1c_1+a'b_2c_2+b_1'ad_1+b_1'b_2'ad_1+b_2'a'd_2+b_1'b_2'a'd_2 && \text{S0,L.7(xiii)}\\ &= ab_1c_1+a'b_2c_2+ab_1'd_1+ab_1'b_2'd_1+a'b_2'd_2+a'b_1'b_2'd_2 && \text{L.7(xiii)} \end{aligned}$$

$$\begin{aligned} s &= a(b_1c_1 + b_1'd_1) + a'(b_2c_2 + b_2'd_2) \\ &= a(b_1c_1 + b_1'd_1 + b_2'b_1'd_1) + a'(b_2c_2 + b_2'd_2 + b_1'b_2'd_2) \\ &\quad \text{by Lemma 7(ii) applied to } b_1'd_1 = b_1'd_1 + b_2'b_1'd_1 \text{ etc.} \\ &= ab_1c_1 + ab_1'd_1 + ab_2'b_1'd_1 + a'b_2c_2 + a'b_2'd_2 + a'b_1'b_2'd_2 \quad \text{S3} \\ &= ab_1c_1 + ab_1'd_1 + a'b_2c_2 + ab_2'b_1'd_1 + a'b_2'd_2 + a'b_1'b_2'd_2 \quad \text{L.7(xiv)} \\ &= ab_1c_1 + a'b_2c_2 + ab_1'd_1 + ab_2'b_1'd_1 + a'b_2'd_2 + a'b_1'b_2'd_2 \quad \text{L.7(xiv)} \\ &= ab_1c_1 + a'b_2c_2 + ab_1'd_1 + ab_1'b_2'd_1 + a'b_2'd_2 + a'b_1'b_2'd_2 \quad \text{L.7(xiii)} \end{aligned}$$

(iii) We get:

$$\begin{aligned} q^{\mathbf{A}^{a-}}(a,b,d) &= ab + a'd \\ &= q^{\mathbf{A}}(q^{\mathbf{A}}(a,b,0), q^{\mathbf{A}}(a,b,0), q^{\mathbf{A}}(q^{\mathbf{A}}(a,0,1),d,0)) \\ &= q^{\mathbf{A}}(q^{\mathbf{A}}(a,b,0), q^{\mathbf{A}}(a,b,0), q^{\mathbf{A}}(a,0,d)) && \text{L.1} \\ &= q^{\mathbf{A}}(a, q^{\mathbf{A}}(b,b,0), q^{\mathbf{A}}(0,0,d)) && \text{Ax}_3 \\ &= q^{\mathbf{A}}(a,b,d) && \text{Ax}_0, \text{Ax}_5 \end{aligned}$$

and similarly:

$$a +^{\mathbf{B}^{-a}} b =_{Def.} aa +^{\mathbf{B}} a'b =_{(S0)} a +^{\mathbf{B}} a'b =_{L.7(vi)} a +^{\mathbf{B}} b$$

$$a \cdot^{\mathbf{B}^{-a}} b =_{Def.} a \cdot^{\mathbf{B}} b + a' \cdot^{\mathbf{B}} 0 =_{(S2)} a \cdot^{\mathbf{B}} b + 0 =_{(S1)} a \cdot^{\mathbf{B}} b$$

$$a'^{\mathbf{B}^{-a}} =_{Def.} a0 + a'^{\mathbf{B}}1 =_{(S2)} 0 + a'^{\mathbf{B}}1 =_{(S1)} a'^{\mathbf{B}}1 =_{L.7(iv)} a'''^{\mathbf{B}} = a'^{\mathbf{B}},$$

since $a''' = (a1)' = a' + 1' = a' + 0 = a$. □

Theorem 10 immediately implies that $\mathcal{NBA}$ is generated by $\mathbf{3}'^{a}$ (cf. Examples 2 and 3), i.e. the algebra whose operations are specified by the following tables:

′	**0**	**2**	**1**
	1	0	0

+	**0**	**2**	**1**
0	0	2	1
2	2	2	2
1	1	1	1

·	**0**	**2**	**1**
0	0	0	0
2	0	2	1
1	0	2	1

In full analogy with Theorem 9, the previous result can be likewise extended to a generic idempotent semi-Boolean-like variety.

Theorem 11 *Every idempotent semi-Boolean-like variety is term equivalent to a variety of noncommutative Boolean algebras with additional regular operations.*

Proof In the light of Theorem 10, our proof is almost complete. Let now $\mathcal{V}$ be an idempotent semi-Boolean-like variety and let $a, b, c \in \mathbf{A} \in \mathcal{V}$. We prove that $\mathbf{A}$ satisfies S8, assuming w.l.g. that g is a binary operation.

$$\begin{aligned} g(ab + a'c, d) &= g(q(ab, ab, a'c), d) && \text{Def.} \\ &= g(q(a, q(b,b,0), q(0,0,c)), d) && \text{Ax}_3, \text{L.1} \\ &= g(q(a,b,c), d) && \text{Ax}_0, \text{Ax}_5 \\ &= q(a, g(b,d), g(c,d)) && \text{Ax}_3 \end{aligned}$$

$$\begin{aligned} a\cdot g(b,d)+a'\cdot g(c,d) &= q(q(a,g(b,d),0),q(a,g(b,d),0),q(a',g(c,d),0)) && \text{Def.}\\ &= q(q(a,g(b,d),0),q(a,g(b,d),0),q(a,0,g(c,d))) && \text{L.1}\\ &= q(a,q(g(b,d),g(b,d),0),q(0,0,g(c,d))) && \text{Ax}_3\\ &= q(a,g(b,d),g(c,d)). && \text{Ax}_0,\text{Ax}_5 \end{aligned}$$

In the opposite direction, let $\mathcal{V}$ be a variety of NBAs with additional regular operations and let $a,b,c\in\mathbf{A}\in\mathcal{V}$. We prove that $\mathbf{A}$ satisfies Ax_3, assuming once more that g is a binary operation.

$$\begin{aligned} & g(q(a,b_1,c_1),q(a,b_2,c_2))\\ &= g(ab_1+a'c_1,ab_2+a'c_2)\\ &= a\cdot g(b_1,ab_2+a'c_2)+a'\cdot g(c_1,ab_2+a'c_2) && \text{S8}\\ &= a(a\cdot g(b_1,b_2)+a'\cdot g(b_1,c_2))+a'(a\cdot g(c_1,b_2)+a'\cdot g(c_1,c_2)) && \text{S8}\\ &= aa\cdot g(b_1,b_2)+aa'\cdot g(b_1,c_2)+a'a\cdot g(c_1,b_2)+a'a'\cdot g(c_1,c_2) && \text{S3}\\ &= aa\cdot g(b_1,b_2)+0\cdot g(b_1,c_2)+0\cdot g(c_1,b_2)+a'a'\cdot g(c_1,c_2) && \text{S5}\\ &= aa\cdot g(b_1,b_2)+a'a'\cdot g(c_1,c_2) && \text{S1, S2}\\ &= a\cdot g(b_1,b_2)+a'\cdot g(c_1,c_2) && \text{S0}\\ &= q(a,g(b_1,b_2),g(c_1,c_2)). \end{aligned}$$

□

We make a note of another immediate consequence of Theorems 10 and 11.

Corollary 1 *Let* $\mathbf{A}$ *be an* NBA. *The following conditions are equivalent:*

(i) $\mathbf{A}$ *is a Boolean algebra;*

(ii) $\mathbf{A}$ *satisfies* $x1\approx x$;

(iii) $\mathbf{A}^-\in\mathcal{B}l\mathcal{A}_0$.

Proof (ii ⇔ i) Let the equation $x1\approx x$ be satisfied. Then, we have:

(a) Commutativity of the product: $ab=ab1=_{L.7.xiii}ba1=ba$.

(b) $(b+1=1)$:

$$b+1=(b+1)1=_{(S3)}b1+1=_{L.7.iii}1.$$

(c) $a+a'=a'+a=a+1=1$ by (b) and S6.

(d) Distributivity of the sum with respect to the product:

$$\begin{aligned} (a+b)(a+d) &= (a+b)a+(a+b)d && \text{S3}\\ &= aa+ba+ad+bd && \text{S3}\\ &= a+ab+ad+bd && \text{S0,(a)}\\ &= a1+ab+ad+bd\\ &= a(1+b+d)+bd && \text{S3}\\ &= a1+bd=a+bd && \text{S1} \end{aligned}$$

(e) $(a''=a)$: $a''=_{L.7.iv}a1=a$.

(f) Commutativity of the sum: $a+b=(a+b)''=(a'b')'=_{(a)}(b'a')'=(b+a)''=b+a$.

(ii ⇔ iii) Since $a1=q(a,1,0)=c(a)$, then we have $c(a)=a$ (that is, $(A;q,1,0)$ is a BlA) iff $a1=a$. □

Acknowledgements We thank the referee for his valuable suggestions.

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