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Common Fixed Point Theorems in a Complete 2-metric Space

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Abstract

In the present paper, we establish a common fixed point theorem for four self-mappings of a complete 2-metric space using the weak commutativity condition and A-contraction type condition and then extend the theorem for a class of mappings.

Key words: fixed point, common fixed point, 2-metric space, completeness

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1 Introduction

In 1981, D. Delbosco [4] gave an unified approach for different contractive mappings to prove the fixed point theorem by considering the set $\mathcal{F}$ of all continuous functions $g\colon [0,+\infty)^3 \to [0,\infty)$ satisfying the following conditions:

(g-1): $g(1,1,1) = h < 1$

(g-2): if $u, v \in [0,\infty)$ are such that $u \le g(v,v,u)$ or, $u \le g(v,u,v)$ or, $u \le g(u,v,v)$; then $u \le hv$.

Recently Akram et al. [1] have modified the above concept slightly and introduced a general class of contractions called A-contraction which is a proper superclass of Kannan's contraction [8], Bianchini's contraction [2] and Reich's contraction [11].

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1.1 A-contraction

Let a nonempty set A consisting of all functions $\alpha\colon R_+^3 \to R_+$ satisfying

(i) α is continuous on the set R_+^3 of all triplets of nonnegative reals(with respect to the Euclidean metric on R^3).

(ii) $a \le kb$ for some $k \in [0,1)$ whenever $a \le \alpha(a,b,b)$ or $a \le \alpha(b,a,b)$ or $a \le \alpha(b,b,a)$, for all a, b.

Definition 1.1 A self map T on a metric space X is said to be A-contraction if it satisfies the condition:

$$d(Tx,Ty) \le \alpha\left(d(x,y), d(x,Tx), d(y,Ty)\right) \tag{1.1}$$

for all $x, y \in X$ and some $\alpha \in A$.

Here we prove a common fixed point theorem for two pairs of weakly commuting mappings using the idea of A-contraction and then extend the theorem for a family of self-mappings in a 2-metric space. Before proving our main theorem we need to state some preliminary ideas and definitions of weakly commuting mappings in a 2-metric space.

2 Preliminaries

In sixties, S. Gähler ([6]–[7]) introduced the concept of 2-metric space. Since then a number of mathematician have been investigating the different aspects of fixed point theory in the setting of 2-metric space.

2.1 2-metric space

Let X be a non empty set. A real valued nonnegative function d on $X \times X \times X$ is said to be a 2-metric on X if

(I) given distinct elements x,y of X, there exists an element z of X such that $d(x,y,z) \neq 0$

(II) $d(x,y,z) = 0$ when at least two of x, y, z are equal,

(III) $d(x,y,z) = d(x,z,y) = d(y,z,x)$ for all x, y, z in X, and

(IV) $d(x,y,z) \le d(x,y,w) + d(x,w,z) + d(w,y,z)$ for all x, y, z, w in X.

When d is a 2-metric on X, then the ordered pair (X,d) is called a 2-metric space.

A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if for each $u \in X$, $\lim d(x_n, x_m, u) = 0$ as $n, m \to \infty$.

A sequence $\{x_n\}$ in X is convergent to an element $x \in X$ if for each $u \in X$, $\lim_{n\to\infty} d(x_n, x, u) = 0$

A complete 2-metric space is one in which every Cauchy sequence in X converges to an element of X.

In 1984, M. D. Khan [9] in his doctoral thesis, defined weakly commuting mappings in a 2-metric space as follows.

Definition 2.1 Let S and T be two mappings from a 2-metric space (X, d) into itself. Then a pair of mappings (S, T) is said to be weakly commuting on x, if $d(STx, TSx, u) \leq d(Tx, Sx, u)$ for all $u \in X$.

Note that a commuting pair (S, T) on a 2-metric space (X, d) is weakly commuting, but the converse is not true (see [10]). On the otherhand Cho–Khan–Singh [3] have proved some common fixed point theorems for weakly commuting self-mappings in a 2-metric space. Here we shall prove some common fixed point theorems in 2-metric space in a more generalised conditions.

3 Main results

Theorem 3.1 *Let I, J, S and T be four self mappings of a complete 2-metric space (X, d) satisfying*

$$I(X) \subset T(X) \quad \text{and} \quad J(X) \subset S(X). \tag{3.1}$$

For $\alpha \in A$ and for all $x, y, u \in X$

$$d(Ix, Jy, u) \leq \alpha(d(Sx, Ty, u), d(Sx, Ix, u), d(Ty, Jy, u)). \tag{3.2}$$

If one of I, J, S and T is continuous and if I and J weakly commute with S and T respectively, then I, J, S and T have a unique common fixed point z in X.

Proof Let x_0 be an arbitrary element of X. We define $Ix_{2n+1} = y_{2n+2}$, $Tx_{2n} = y_{2n}$ and $Jx_{2n} = y_{2n+1}$, $Sx_{2n+1} = y_{2n+1}$; $n = 1, 2, \ldots$ Taking $x = x_{2n+1}$ and $y = x_{2n}$ in (3.2) we have

$$d(Ix_{2n+1}, Jx_{2n}, u) \leq$$
$$\leq \alpha(d(Sx_{2n+1}, Tx_{2n}, u), d(Sx_{2n+1}, Ix_{2n+1}, u), d(Tx_{2n}, Jx_{2n}, u))$$

or,

$$d(y_{2n+2}, y_{2n+1}, u) \leq \alpha(d(y_{2n+1}, y_{2n}, u), d(y_{2n+1}, y_{2n+2}, u), d(y_{2n}, y_{2n+1}, u)).$$

So by axiom (ii) of function α,

$$d(y_{2n+1}, y_{2n+2}, u) \leq k.d(y_{2n}, y_{2n+1}, u) \quad \text{where } k \in [0, 1) \tag{3.3}$$

Similarly by putting $x = x_{2n-1}$ and $y = x_{2n}$ in (3.2) we get

$$d(Ix_{2n-1}, Jx_{2n}, u) \leq$$
$$\leq \alpha(d(Sx_{2n-1}, Tx_{2n}, u), d(Sx_{2n-1}, Ix_{2n-1}, u), d(Tx_{2n}, Jx_{2n}, u))$$

or,

$$d(y_{2n}, y_{2n+1}, u) \leq \alpha(d(y_{2n-1}, y_{2n}, u), d(y_{2n-1}, y_{2n}, u), d(y_{2n}, y_{2n+1}, u)).$$

So by axiom (ii) of function α,

$$d(y_{2n}, y_{2n+1}, u) \leq k.d(y_{2n-1}, y_{2n}, u) \quad \text{where } k \in [0,1) \tag{3.4}$$

So by (3.3) and (3.4) we get

$$d(y_{2n+1}, y_{2n+2}, u) \leq k \cdot d(y_{2n}, y_{2n+1}, u) \leq k^2 \cdot d(y_{2n-1}, y_{2n}, u).$$

Proceeding in this way

$$d(y_{2n+1}, y_{2n+2}, u) \leq k^{2n+1} \cdot d(y_0, y_1, u)$$

and

$$d(y_{2n}, y_{2n+1}, u) \leq k^{2n} \cdot d(y_0, y_1, u).$$

So in general

$$d(y_n, y_{n+1}, u) \leq k^n \cdot d(y_0, y_1, u). \tag{3.5}$$

Then using property (IV) of 2-metric space we get

$$\begin{aligned} d(y_n, y_{n+2}, u) &\leq d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, u) + d(y_{n+1}, y_{n+2}, u) \\ &\leq d(y_n, y_{n+2}, y_{n+1}) + \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u). \end{aligned} \tag{3.6}$$

Here we consider two possible cases to show that $d(y_n, y_{n+2}, y_{n+1}) = 0$.

Case I. $n = \text{even} = 2m$ (say)

Therefore

$$\begin{aligned} d(y_n, y_{n+2}, y_{n+1}) &= d(y_{2m}, y_{2m+2}, y_{2m+1}) \\ &= d(y_{2m+2}, y_{2m+1}, y_{2m}) \\ &= d(Ix_{2m+1}, Jx_{2m}, y_{2m}) \\ &\leq \alpha(d(Sx_{2m+1}, Tx_{2m}, y_{2m}), d(Sx_{2m+1}, Ix_{2m+1}, y_{2m}), \\ &\quad d(Tx_{2m}, Jx_{2m}, y_{2m})) \\ &= \alpha(d(y_{2m+1}, y_{2m}, y_{2m}), d(y_{2m+1}, y_{2m+2}, y_{2m}), \\ &\quad d(y_{2m}, y_{2m+1}, y_{2m})) \\ &= \alpha(0, d(y_{2m+1}, y_{2m+2}, y_{2m}), 0). \end{aligned}$$

So by axiom (ii) of function α,

$$d(y_n, y_{n+2}, y_{n+1}) = d(y_{2m}, y_{2m+2}, y_{2m+1}) \leq k \cdot 0 = 0 \quad \text{where } k \in [0,1)$$

which implies $d(y_n, y_{n+2}, y_{n+1}) = 0$.

Case II. $n = \text{odd} = 2m+1$ (say)

Therefore

$$\begin{aligned} d(y_n, y_{n+2}, y_{n+1}) &= d(y_{2m+1}, y_{2m+3}, y_{2m+2}) \\ &= d(y_{2m+3}, y_{2m+2}, y_{2m+1}) \\ &= d(Jx_{2m+2}, Ix_{2m+1}, y_{2m+1}) \\ &\leq \alpha\left(d(Sx_{2m+1}, Tx_{2m+2}, y_{2m+1}),\right. \\ &\quad \left. d(Sx_{2m+1}, Ix_{2m+1}, y_{2m+1}), d(Tx_{2m+2}, Jx_{2m+2}, y_{2m+1})\right) \\ &= \alpha\left(d(y_{2m+1}, y_{2m+2}, y_{2m+1}), d(y_{2m+1}, y_{2m+2}, y_{2m+1}),\right. \\ &\quad \left. d(y_{2m+2}, y_{2m+3}, y_{2m+1})\right) \\ &= \alpha\left(0, 0, d(y_{2m+2}, y_{2m+3}, y_{2m+1})\right). \end{aligned}$$

Then by axiom (ii) of function α,

$$d(y_n, y_{n+2}, y_{n+1}) = d(y_{2m+1}, y_{2m+3}, y_{2m+2}) \leq k \cdot 0 = 0 \quad \text{where } k \in [0,1)$$

So in either cases $d(y_n, y_{n+2}, y_{n+1}) = 0$. Therefore from (3.6) we have

$$d(y_n, y_{n+2}, u) \leq \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u).$$

Proceeding in the same fashion we have for any $p > 0$,

$$d(y_n, y_{n+p}, u) \leq \sum_{r=0}^{p-1} d(y_{n+r}, y_{n+r+1}, u).$$

Then by (3.5) we get

$$d(y_n, y_{n+p}, u) \leq \frac{k^n}{1-k} d(y_0, y_1, u) \to 0 \quad \text{as } n \to \infty,\ p > 0 \text{ and } k \in [0,1).$$

Hence $\{y_n\}$ is a Cauchy sequence. Then by completeness of X, $\{y_n\}$ converges to a point $z \in X$ i.e. $y_n \to z \in X$ as $n \to \infty$. Since $\{y_n\}$ is a Cauchy sequence and taking limit as $n \to \infty$, we get $Ix_{2n} = Tx_{2n+1} \to z$, $Jx_{2n-1} = Sx_{2n} \to z$ and also $Jx_{2n+1} \to z$.

Next suppose that S is continuous. Then $\{SIx_{2n}\}$ converges to Sz. Then by property (IV) of 2-metric space, we have

$$\begin{aligned} d(ISx_{2n}, Sz, u) &\leq d(ISx_{2n}, Sz, SIx_{2n}) + d(ISx_{2n}, SIx_{2n}, u) + d(SIx_{2n}, Sz, u) \\ &\leq d(ISx_{2n}, Sz, SIx_{2n}) + d(Sx_{2n}, Ix_{2n}, u) + d(SIx_{2n}, Sz, u), \end{aligned}$$

since I and S weakly commute.

Letting $n \to \infty$, it follows that $\{ISx_{2n}\}$ converges to Sz. Again by using (3.2) we have

$$\begin{aligned} &d(ISx_{2n}, Jx_{2n+1}, u) \leq \\ &\leq \alpha\left(d\left(S^2x_{2n}, Tx_{2n+1}, u\right), d\left(S^2x_{2n}, ISx_{2n}, u\right), d(Tx_{2n+1}, Jx_{2n+1}, u)\right). \end{aligned}$$

Since α is continuous, taking limit as $n \to \infty$ we get

$$d(Sz, z, u) \leq \alpha(d(Sz, z, u), d(Sz, Sz, u), d(z, z, u))$$

implies

$$d(Sz, z, u) \leq \alpha(d(Sz, z, u), 0, 0).$$

So by axiom (ii) of function α,

$$d(Sz, z, u) \leq k \cdot 0 = 0 \quad \text{which gives } Sz = z. \tag{3.7}$$

Again using the inequality (3.2) we have

$$d(Iz, Jx_{2n+1}, u) \leq \alpha(d(Sz, Tx_{2n+1}, u), d(Sz, Iz, u), d(Tx_{2n+1}, Jx_{2n+1}, u)).$$

Passing limit as $n \to \infty$ we get

$$d(Iz, z, u) \leq \alpha(d(Sz, z, u), d(z, Iz, u), d(z, z, u))$$

implies

$$d(Iz, z, u) \leq \alpha(0, d(z, Iz, u), 0).$$

Then by axiom (ii) of function α,

$$d(Iz, z, u) \leq k \cdot 0 = 0 \quad \text{which gives } Iz = z. \tag{3.8}$$

Since $I(X) \subset T(X)$, there exists a point $z' \in X$ such that $Tz' = z = Iz$, so by (3.2) we have

$$\begin{aligned} d(z, Jz', u) &= d(Iz, Jz', u) \\ &\leq \alpha(d(Sz, Tz', u), d(Sz, Iz, u), d(Tz', Jz', u)) \\ &= \alpha(d(z, z, u), d(z, z, u), d(z, Jz', u)) \\ &= \alpha(0, 0, d(z, Jz', u)). \end{aligned}$$

So by axiom (ii) of function α,

$$d(z, Jz', u) \leq k \cdot 0 = 0 \quad \text{which implies } Jz' = z.$$

As J and T weakly commute

$$d(JTz', TJz', u) \leq d(Tz', Jz', u) = 0$$

which gives $JTz' = TJz'$ implies

$$Jz = JTz' = TJz' = Tz. \tag{3.9}$$

Thus from (3.2) we have

$$\begin{aligned} d(z, Tz, u) &= d(Iz, Jz, u) \\ &\leq \alpha(d(Sz, Tz, u), d(Sz, Iz, u), d(Tz, Jz, u)) \\ &= \alpha(d(z, Tz, u), 0, 0). \end{aligned}$$

So by axiom (ii) of function α,

$$d(z, Tz, u) \leq k \cdot 0 = 0 \quad \text{which implies } Tz = z. \tag{3.10}$$

So by (3.7),(3.8),(3.9) and (3.10) we conclude that z is a common fixed point of I, J, S and T.

For uniqueness, Let w be another common fixed point in X such that

$$Iz = Jz = Sz = Tz = z \quad \text{and} \quad Iw = Jw = Sw = Tw = w.$$

Then by (3.2) we have

$$\begin{aligned} d(w, z, u) &= d(Iw, Jz, u) \\ &\leq \alpha(d(Sw, Tz, u), d(Sw, Iw, u), d(Tz, Jz, u)) \\ &= \alpha(d(w, z, u), d(w, w, u), d(z, z, u)) \\ &= \alpha(d(w, z, u), 0, 0). \end{aligned}$$

So by axiom (ii) of function α,

$$d(w, z, u) \leq k \cdot 0 = 0 \quad \text{which implies } w = z.$$

So uniqueness of z is proved.

The same result holds if any one of I, J and T is continuous. □

Corollary 3.2 *Let S, T, I and J be four self mappings of a complete 2-metric space (X, d) satisfying*

$$I(X) \subset T(X) \text{ and } J(X) \subset S(X) \tag{3.11}$$

$$d(Ix, Jy, u) \leq c \cdot \max\{d(Sx, Ty, u), d(Sx, Ix, u), d(Ty, Jy, u)\} \tag{3.12}$$

for all x, y, u in X, where $0 \leq c < 1$.

If one of S, T, I and J is continuous and if I and J weakly commute with S and T respectively, then I, J, S and T have a unique common fixed point z in X.

This result is a 2-metric analogue of the theorem of B. Fisher [5].

For any $f\colon (X, d) \to (X, d)$ we denote $F_f = \{x \in X\colon x = f(x)\}$.

Lemma 3.3 *Let I, J, S and T be four self mappings of a complete 2-metric space (X, d). If the inequality (3.2) holds for $\alpha \in A$ and for all $x, y, u \in X$. Then $(F_S \cap F_T) \cap F_I = (F_S \cap F_T) \cap F_J$.*

Proof Let $x \in (F_S \cap F_T) \cap F_I$. Then by(3.2)

$$\begin{aligned} d(x, Jx, u) &= d(Ix, Jx, u) \\ &\leq \alpha(d(Sx, Tx, u), d(Sx, Ix, u), d(Tx, Jx, u)) \\ &= \alpha(0, 0, d(x, Jx, u)). \end{aligned}$$

So by axiom (ii) of function α,

$$d(x, Jx, u) \leq k \cdot 0 = 0 \quad \text{implies } x = Jx.$$

Thus

$$(F_S \cap F_T) \cap F_I \subset (F_S \cap F_T) \cap F_J.$$

Similarly we have

$$(F_S \cap F_T) \cap F_J \subset (F_S \cap F_T) \cap F_I$$

and so $(F_S \cap F_T) \cap F_I = (F_S \cap F_T) \cap F_J$ □

Theorem 3.4 *Let S, T and $\{I_n\}_{n\in N}$ be mappings from a complete 2-metric space (X, d) into itself satisfying*

$$I_1(X) \subset T(X) \text{ and } I_2(X) \subset S(X). \tag{3.13}$$

For $\alpha \in A$ and for all $x, y, u \in X$,

$$d(I_n x, I_{n+1} y, u) \leq \alpha(d(Sx, Ty, u), d(Sx, I_n x, u), d(Ty, I_{n+1} y, u)) \tag{3.14}$$

holds for all $n \in N$. If one of S, T, I_1 and I_2 is continuous and if I_1 and I_2 weakly commute with S and T respectively, then S, T and $\{I_n\}_{n\in N}$ have a unique common fixed point z in X.

Proof By Theorem 3.1, S, T, I_1 and I_2 have a unique common fixed point z in X. Now z is a unique common fixed point of S, T, I_1 and also by Lemma 3.3, $(F_S \cap F_T) \cap F_{I_1} = (F_S \cap F_T) \cap F_{I_2}$, z is a common fixed point of S, T, I_2. Also z is unique common fixed point of S, T, I_2. If not, let w be another common fixed point of S, T, I_2. Then by (3.14)

$$\begin{aligned} d(z, w, u) &= d(I_1 z, I_2 w, u) \\ &\leq \alpha(d(Sz, Tw, u), d(Sz, I_1 z, u), d(Tw, I_2 w, u)) \\ &= \alpha(d(z, w, u), d(z, z, u), d(w, w, u)) \\ &= \alpha(d(z, w, u), 0, 0). \end{aligned}$$

So by axiom (ii) of function α,

$$d(z, w, u) \leq k \cdot 0 = 0 \quad \text{implies } z = w.$$

In the similar manner we can show that z is a unique common fixed point of S, T and I_3. Continuing in this way, we arrive at desired result. □

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