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The Rings Which Can Be Recovered by Means of the Difference*

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Abstract

It is well known that to every Boolean ring $\mathcal{R}$ can be assigned a Boolean algebra $\mathcal{B}$ whose operations are term operations of $\mathcal{R}$. Then a symmetric difference of $\mathcal{B}$ together with the meet operation recover the original ring operations of $\mathcal{R}$. The aim of this paper is to show for what a ring $\mathcal{R}$ a similar construction is possible. Of course, we do not construct a Boolean algebra but only so-called lattice-like structure which was introduced and treated by the authors in a previous paper. In particular, we reached interesting results if the characteristic of the ring $\mathcal{R}$ is either an odd natural number or a power of 2.

Key words: Boolean ring, commutative ring, lattice-like structure, difference

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Having a Boolean ring $\mathcal{R} = (R; +, \cdot, 0, 1)$ the induced Boolean algebra $\mathcal{B}(\mathcal{R}) = (R; \vee, \wedge, {}', 0, 1)$ can be established by

$$x \vee y = x + y + xy, \quad x \wedge y = xy, \quad x' = 1 + x,$$

see [1]. Also conversely, having a Boolean algebra $\mathcal{B} = (B; \vee, \wedge, {}', 0, 1)$ we can define the so-called symmetric difference $x + y = (x \wedge y') \vee (x' \wedge y)$ and, using this, the induced Boolean ring $\mathcal{R}(\mathcal{B}) = (B; +, \cdot, 0, 1)$ can be recovered as follows:

- $x + y$ in $\mathcal{R}(\mathcal{B})$ is equal to the symmetrical difference of $\mathcal{B}$;
- $x \cdot y$ in $\mathcal{R}(\mathcal{B})$ is equal to the meet operation $\wedge$ of $\mathcal{B}$.

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It was already shown by several authors that a similar construction can be derived also for orthomodular lattices (see [6, 7, 8, 9, 10, 11, 12]), for ortholattices (see [2, 3]), or for pseudocomplemented semilattices, [4]. The construction was generalized for bounded lattices with an antitone involution in [12]. However, the ring-like structures induced by these lattices are rather far from rings.

Hence, we involved another approach in [5] in order to establish a certain lattice-like structure to a given ring such that the original ring can be recovered by means of the difference. Of course, this is not possible for every ring but it appeared in [5] that the construction works for commutative unitary rings of characteristic 2 satisfying the identity $x^{p+2} = x^p$ for a natural number p (if $p = 1$, then the ring is Boolean and the assigned lattice-like structure is a Boolean algebra).

The aim of this paper is to extend this approach to a broader class of rings which contains some rings of residue classes and rings of characteristic different from 2.

All the rings considered in the paper are *commutative* (i.e. satisfying the identity $x \cdot y = y \cdot x$) and *unitary* (i.e. having an element 1 with $x \cdot 1 = x$).

Let p be a given natural number and $\mathcal{R} = (R; +, \cdot, 0, 1)$ a ring. Let us agree in the following notation:

- $x' = 1 + x$;
- $x^* = 1 - x$;
- $x \wedge y = x \cdot y$;
- $x \vee y = x + y + x^p \cdot y^p$.

The induced algebra $\mathcal{L}(\mathcal{R}) = (R; \vee, \wedge, ', ^*, 0, 1)$ will be referred to as a *lattice-like structure* induced by $\mathcal{R}$ and the term operation

- $x \oplus y = (x \wedge y') \vee (x^* \wedge y)$

as a *difference*. We will not use the name symmetric difference because $x \oplus y$ need not be equal to $y \oplus x$ in general. However, if $\mathcal{R}$ is recovered by the induced lattice-like structure, i.e. $x + y = x \oplus y$, then $x \oplus y = y \oplus x$, and hence it is symmetric.

In contrast to [5], we are not interested in the properties of the induced lattice-like structure $\mathcal{L}(\mathcal{R})$ but only in the case when $\mathcal{R}$ can be recovered from $\mathcal{L}(\mathcal{R})$ by means of the difference $\oplus$ (since the second operation "$\cdot$" coincides with "$\wedge$" of $\mathcal{L}(\mathcal{R})$). In other words, we search for rings where $x + y = x \oplus y$. To show that this is possible also for rings which are not Boolean, let us give the following.

Example 1 Let $\mathcal{R} = (\{0, 1, 2, 3\}, +, \cdot)$ be a ring whose operations are determined by the tables

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

It is immediate that $\mathcal{R}$ is commutative, unitary and of characteristic 2. It satisfies the identity $x^4 = x^2$ since $0^4 = 0 = 0^2$, $1^4 = 1 = 1^2$, $2^4 = 0 = 2^2$ and $3^4 = 1 = 3^2$. Moreover, due to the foregoing definitions,

$$x \oplus y = (x \wedge y') \vee (x^* \wedge y) = x + y$$

but $\mathcal{R}$ is not Boolean, because e.g. $2^2 = 0 \neq 2$. Let us note that this ring $\mathcal{R}$ is isomorphic to the polynomial ring $\mathcal{Z}_2[x]/(x^2)$, where (x^2) is the principal ideal of $\mathcal{Z}_2[x]$ generated by x^2.

Now, we are focused on the rings of residue classes $\mathcal{Z}_p$.

Theorem 2 *Let $\mathcal{Z}_p$ be the ring of residue classes modulo p with $p = 2^k$ for some $k \in \mathbb{N}$. Then $x + y = x \oplus y$, i.e. $\mathcal{Z}_p$ can be recovered from the induced lattice-like structure.*

Proof Since $x' = 1 + x$, $x^* = 1 - x$, $x \wedge y = x \cdot y$ and $x \vee y = x + y + x^p \cdot y^p$, we get

$$\begin{aligned} x \oplus y &= (x \wedge y') \vee (x^* \wedge y) \\ &= x \cdot (1 + y) + (1 - x) \cdot y + x^p \cdot (1 + y)^p \cdot (1 - x)^p \cdot y^p \\ &= x + x \cdot y + y - x \cdot y + (x^p \cdot (1 - x)^p) \cdot (y^p \cdot (1 + y)^p) \\ &= x + y + (x \cdot (1 - x))^p \cdot (y \cdot (1 + y))^p \end{aligned}$$

However, for every $y \in \mathcal{Z}_p$ we have that $y \cdot (1 + y)$ is an even number (mod p), e.g. $y \cdot (1 + y) = 2a$. Since $p = 2^k$, we have $2^k \equiv 0 \mod p$. Thus

$$(y \cdot (1 + y))^p = (2a)^p = 2^p \cdot a^p = 2^k \cdot 2^{2^k - k} \cdot a^p \equiv 0 \cdot 2^{2^k - k} \cdot a^p \equiv 0 \mod p$$

where we used the fact that $k < 2^k$ for every $k \in \mathbb{N}$. Hence

$$(x \cdot (1 - x))^p \cdot (y \cdot (1 + y))^p = 0$$

for every $x, y \in \mathcal{Z}_p$ which implies $x + y = x \oplus y$. □

If $p \neq 2^k$ for some $k \in \mathbb{N}$, then the addition in $\mathcal{Z}_p$ need not be equal to the difference, see the following.

Example 3 Consider the ring $\mathcal{Z}_3$ of residue classes modulo 3. For $x = 2$ we have $(x \cdot (1 - x))^3 = (2 \cdot 2)^3 = 1^3 = 1$ and for $y = 1$ we have $(y \cdot (1 + y))^3 = (1 \cdot 2)^3 = 2^3 = 2$. Together we get $(x \cdot (1 - x))^p \cdot (y \cdot (1 + y))^p = 1 \cdot 2 = 2 \neq 0$.

The result of Example 3 can be extended for each ring of an odd characteristic.

Theorem 4 *Let p be an odd natural number and $\mathcal{R} = (R; +, \cdot, 0, 1)$ a ring of characteristic p. Then $x \oplus y \neq x + y$.*

Proof Assume that p is an odd natural number and let $char(\mathcal{R}) = p$. Take $x = -1$ and $y = 1$. Since p is odd, we have $-1 \neq 1$. Then for

$$x \oplus y = x + y + (x \cdot (1-x))^p \cdot (y \cdot (1+y))^p$$

we have $(x \cdot (1-x))^p \cdot (y \cdot (1+y))^p = -4^p \neq 0$, thus $x \oplus y \neq x + y$. □

On the contrary, if the characteristic of $\mathcal{R}$ is equal to 2, then we can easily characterize rings for which $x \oplus y = x + y$.

Remark 5 Let us note that if $\mathcal{R}$ is of characteristic 2, then

$$x' = 1 + x = 1 - x = x^*,$$

thus $\oplus$ is defined formally in the same way as for Boolean rings.

Theorem 6 *Let $p = 2^k$ for some $k \in \mathbb{N}$ and $\mathcal{R} = (R; +, \cdot, 0, 1)$ be a ring of characteristic 2. Then $x \oplus y = x + y$ if and only if $\mathcal{R}$ satisfies the identity*

$$\left(x^{2p} + x^p\right)\left(y^{2p} + y^p\right) = 0.$$

Proof Since $\mathrm{char}(\mathcal{R}) = 2$, we get $-x = x$ for each $x \in R$. Thus

$$x \oplus y = x + y + (x \cdot (1+x))^p \cdot (y \cdot (1+y))^p.$$

Moreover, $p = 2^k$, which implies $(1+x)^p = 1 + x^p$. Altogether, $x \oplus y = x + y$ if and only if

$$x^p(1+x^p)y^p(1+y^p) = 0$$

in $\mathcal{R}$ which is if and only if $(x^{2p} + x^p)(y^{2p} + y^p) = 0$ in $\mathcal{R}$. □

Remark 7 If $\mathcal{R} = (R; +, \cdot, 0, 1)$ is of characteristic 2 and $p = 2^k$, then the identity

$$x^p(1+x)^p = x^{2p} + x^p = 0,$$

i.e. $x^{2p} = x^p$, yields the identity of Theorem 6. It is a question if the identity of Theorem 6 can be replaced by this simplier one. The following example shows that this is not possible.

Example 8 Let $\mathcal{R} = (\{0, 1, \ldots, 31\}, +, \cdot)$ be a ring whose operations are determined by the tables 1 and 2. This ring is of characteristic 2, clearly, it is not Boolean, moreover, it satifies the identity

$$\left(x^8 + x^4\right)\left(y^8 + y^4\right) = 0,$$

but $x^8 = x^4$ does not hold in $\mathcal{R}$ because e.g. $2^8 = 0 \neq 16 = 2^4$.

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	1	0	3	2	6	7	4	5	13	15	14	12	11	8	10	9	20	31	24	30	16	26	29	27	18	28	21	23	25	22	19	17
2	2	3	0	1	5	4	7	6	10	12	8	15	9	14	13	11	22	27	25	21	29	19	16	31	28	18	30	17	24	20	26	23
3	3	2	1	0	7	6	5	4	14	11	13	9	15	10	8	12	29	23	28	26	22	30	20	17	25	24	19	31	18	16	21	27
4	4	6	5	7	0	2	1	3	9	8	12	14	10	15	11	13	18	21	16	27	24	17	25	30	20	22	31	19	29	28	23	26
5	5	7	4	6	2	0	3	1	12	10	9	13	8	11	15	14	25	19	22	17	28	27	18	26	29	16	23	21	20	24	31	30
6	6	4	7	5	1	3	0	2	15	13	11	10	14	9	12	8	24	26	20	23	18	31	28	19	16	29	17	30	22	25	27	21
7	7	5	6	4	3	1	2	0	11	14	15	8	13	12	9	10	28	30	29	31	25	23	24	21	22	20	27	26	16	18	17	19
8	8	13	10	14	9	12	15	11	0	4	2	7	5	1	3	6	17	16	21	25	31	18	27	29	26	19	24	22	30	23	28	20
9	9	15	12	11	8	10	13	14	4	0	5	3	2	6	7	1	21	18	17	22	26	16	19	28	31	27	20	25	23	30	29	24
10	10	14	8	13	12	9	11	15	2	5	0	6	4	3	1	7	27	22	19	18	23	25	17	20	30	21	28	16	26	31	24	29
11	11	12	15	9	14	13	10	8	7	3	6	0	1	5	4	2	30	28	23	20	19	29	26	18	27	31	22	24	17	21	16	25
12	12	11	9	15	10	8	14	13	5	2	4	1	0	7	6	3	19	25	27	16	30	22	21	24	23	17	29	18	31	26	20	28
13	13	8	14	10	15	11	9	12	1	6	3	5	7	0	2	4	31	20	26	28	17	24	23	22	21	30	18	29	19	27	25	16
14	14	10	13	8	11	15	12	9	3	7	1	4	6	2	0	5	23	29	30	24	27	28	31	16	19	26	25	20	21	17	18	22
15	15	9	11	12	13	14	8	10	6	1	7	2	3	4	5	0	26	24	31	29	21	20	30	25	17	23	16	28	27	19	22	18
16	16	20	22	29	18	25	24	28	17	21	27	30	19	31	23	26	0	8	4	12	1	9	2	14	6	5	15	10	7	3	11	13
17	17	31	27	23	21	19	26	30	16	18	22	28	25	20	29	24	8	0	9	5	13	4	10	3	15	12	6	2	11	14	7	1
18	18	24	25	28	16	22	20	29	21	17	19	23	27	26	30	31	4	9	0	10	6	8	5	11	1	2	13	12	3	7	14	15
19	19	30	21	26	27	17	23	31	25	22	18	20	16	28	24	29	12	5	10	0	11	2	9	6	14	8	3	4	13	15	1	7
20	20	16	29	22	24	28	18	25	31	26	23	19	30	17	27	21	1	13	6	11	0	15	3	10	4	7	9	14	5	2	12	8
21	21	26	19	30	17	27	31	23	18	16	25	29	22	24	28	20	9	4	8	2	15	0	12	7	13	10	1	5	14	11	3	6
22	22	29	16	20	25	18	28	24	27	19	17	26	21	23	31	30	2	10	5	9	3	12	0	13	7	4	11	8	6	1	15	14
23	23	27	31	17	30	26	19	21	29	28	20	18	24	22	16	25	14	3	11	6	10	7	13	0	12	15	5	1	9	8	4	2
24	24	18	28	25	20	29	16	22	26	31	30	27	23	21	19	17	6	15	1	14	4	13	7	12	0	3	8	11	2	5	10	9
25	25	28	18	24	22	16	29	20	19	27	21	31	17	30	26	23	5	12	2	8	7	10	4	15	3	0	14	9	1	6	13	11
26	26	21	30	19	31	23	17	27	24	20	28	22	29	18	25	16	15	6	13	3	9	1	11	5	8	14	0	7	10	12	2	4
27	27	23	17	31	19	21	30	26	22	25	16	24	18	29	20	28	10	2	12	4	14	5	8	1	11	9	7	0	15	13	6	3
28	28	25	24	18	29	20	22	16	30	23	26	17	31	19	21	27	7	11	3	13	5	14	6	9	2	1	10	15	0	4	8	12
29	29	22	20	16	28	24	25	18	23	30	31	21	26	27	17	19	3	14	7	15	2	11	1	8	5	6	12	13	4	0	9	10
30	30	19	26	21	23	31	27	17	28	29	24	16	20	25	18	22	11	7	14	1	12	3	15	4	10	13	2	6	8	9	0	5
31	31	17	23	27	26	30	21	19	20	24	29	25	28	16	22	18	13	1	15	7	8	6	14	2	9	11	4	3	12	10	5	0

Table 1

·	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
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6	0	6	10	11	18	19	20	23	8	21	2	29	25	15	7	31	16	17	4	5	24	9	27	28	1	12	13	22	14	30	3	26
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9	0	9	17	18	16	8	21	4	0	16	17	4	8	9	18	21	0	0	16	8	9	16	17	18	21	8	21	17	4	18	4	9
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11	0	11	19	20	21	2	29	15	17	4	5	24	27	28	13	14	16	8	9	10	30	18	12	31	3	22	23	25	26	1	6	7
12	0	12	21	22	17	4	25	10	16	8	9	27	18	19	2	5	0	16	17	18	12	8	21	2	25	4	5	9	10	22	27	19
13	0	13	22	23	4	25	15	30	8	9	27	28	19	1	29	6	16	17	18	12	31	21	2	3	26	5	24	10	11	14	7	20
14	0	14	25	26	9	27	7	20	17	18	12	13	2	29	6	30	16	8	21	22	23	4	5	24	28	10	11	19	1	15	31	3
15	0	15	27	28	18	12	31	3	8	21	22	14	5	6	30	20	16	17	4	25	26	9	10	11	13	19	1	2	29	7	23	24
16	0	16	0	16	0	0	16	16	0	0	0	16	0	16	16	16	0	0	0	0	16	0	0	16	16	0	16	0	16	16	16	16
17	0	17	16	8	0	16	17	8	0	0	16	8	16	17	8	17	0	0	0	16	17	0	16	8	17	16	17	16	8	8	8	17
18	0	18	8	21	16	17	4	9	0	16	8	9	17	18	21	4	0	0	16	17	18	16	8	21	4	17	4	8	9	21	9	18
19	0	19	21	2	17	4	5	27	16	8	9	10	18	12	22	25	0	16	17	18	19	8	21	22	5	4	25	9	27	2	10	12
20	0	20	2	29	4	5	24	28	8	9	10	30	12	31	23	26	16	17	18	19	1	21	22	14	6	25	15	27	7	3	11	13
21	0	21	17	4	16	8	9	18	0	16	17	18	8	21	4	9	0	0	16	8	21	16	17	4	9	8	9	17	18	4	18	21
22	0	22	4	25	8	9	27	19	16	17	18	12	21	2	5	10	0	16	8	21	22	17	4	5	27	9	10	18	19	25	12	2
23	0	23	25	15	9	27	28	1	17	18	12	31	2	3	24	11	16	8	21	22	14	4	5	6	7	10	30	19	20	26	13	29
24	0	24	10	30	18	19	1	14	8	21	2	3	25	26	28	13	16	17	4	5	6	9	27	7	20	12	31	22	23	11	29	15
25	0	25	9	27	17	18	12	2	16	8	21	22	4	5	10	19	0	16	17	4	25	8	9	10	12	18	19	21	2	27	22	5
26	0	26	27	7	18	12	13	29	8	21	22	23	5	24	11	1	16	17	4	25	15	9	10	30	31	19	20	2	3	28	14	6
27	0	27	18	12	8	21	22	5	16	17	4	25	9	10	19	2	0	16	8	9	27	17	18	19	22	21	2	4	5	12	25	10
28	0	28	12	31	21	22	14	6	17	4	25	26	10	11	1	29	16	8	9	27	7	18	19	20	23	2	3	5	24	13	15	30
29	0	29	5	24	9	10	30	31	17	18	19	1	22	14	15	7	16	8	21	2	3	4	25	26	11	27	28	12	13	6	20	23
30	0	30	19	1	21	2	3	26	17	4	5	6	27	7	31	23	16	8	9	10	11	18	12	13	29	22	14	25	15	20	24	28
31	0	31	22	14	4	25	26	11	8	9	27	7	19	20	3	24	16	17	18	12	13	21	2	29	15	5	6	10	30	23	28	1

Table 2

References

[1] Birkhoff, G.: Lattice Theory. 3rd edition, *AMS Colloq. Publ.* **25**, Providence, RI, 1979.

[2] Chajda, I.: *Pseudosemirings induced by ortholattices*. Czech. Math. J. **46** (2008), 405–411.

[3] Chajda, I., Eigenthaler, G.: *A note on orthopseudorings and Boolean quasirings*. Österr. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **207** (1998), 83–94.

[4] Chajda, I., Länger, H.: *Ring-like operations in pseudocomplemented semilattices*. Discuss. Math., Gen. Algebra Appl. **20** (2010), 87–95.

[5] Chajda, I., Švrček, F.: *Lattice-like structures derived from rings*. In: Proc. of Salzburg Conference (AAA81), Contributions to General Algebra **20**, *J. Heyn*, Klagenfurt, 2011.

[6] Dorninger, D., Länger, H., Mączyński, M.: *The logic induced by a system of homomorphisms and its various algebraic characterizations*. Demonstratio Math. **30** (1997), 215–232.

[7] Dorninger, D., Länger, H., Mączyński, M.: *On ring-like structures occuring in axiomatic quantum mechanics*. Österr. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **206** (1997), 279–289.

[8] Dorninger, D., Länger, H., Mączyński, M.: *On ring-like structures induced by Mackey's probability function*. Rep. Math. Phys. **43** (1999), 499–515.

[9] Dorninger, D., Länger, H., Mączyński, M.: *Lattice properties of ring-like quantum logics*. Intern. J. Theor. Phys. **39** (2000), 1015–1026.

[10] Dorninger, D., Länger, H., Mączyński, M.: *Concepts of measures on ring-like quantum logics*. Rep. Math. Phys. **47** (2001), 167–176.

[11] Dorninger, D., Länger, H., Mączyński, M.: *Ring-like structures with unique symmetric difference related to quantum logic*. Discuss. Math., Gen. Algebra Appl. **21** (2001), 239–253.

[12] Länger, H.: *Generalizations of the corresspondence between Boolean algebras and Boolean rings to orthomodular lattices*. Tatra Mt. Math. Publ. **15** (1998), 97–105.