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On Equational Theory of Left Divisible Left Distributive Groupoids*

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Abstract

It is an open question whether the variety generated by the left divisible left distributive groupoids coincides with the variety generated by the left distributive left quasigroups. In this paper we prove that every left divisible left distributive groupoid with the mapping $a \mapsto a^2$ surjective lies in the variety generated by the left distributive left quasigroups.

Key words: left distributivity, left idempotency, variety

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Many groupoids that satisfy the left distributivity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z) \tag{LD}$$

satisfy the idempotency

$$x \cdot x = x \tag{I}$$

too. An example of such a left distributive idempotent (LDI) groupoid is a group G with the conjugacy, i.e. the operation $x\hat{\ }y = xyx^{-1}$. It was an open question for a long time whether the groupoids of the group conjugacy (GC) generate all the variety LDI or if there exists an equation that holds in GC and not in LDI. This question was solved independently by D. Larue [7] and A. Drápal, T. Kepka and R. Musílek [3]. Moreover, we have the following characterization:

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Theorem 1 (Joyce [5], Kepka [6], Larue [7]) *The following varieties coincide:*

- *the variety generated by GC;*
- *the variety generated by the left cancellative LDI groupoids;*
- *the variety generated by the left divisible LDI groupoids;*
- *the variety generated by the LDI left quasigroups (i.e. left cancellative left divisible LDI groupoids).*

Does an analogous characterization hold without the idempotency? All left divisible left distributive (LDLD) groupoids satisfy the following identity:

$$(x \cdot x) \cdot y = x \cdot y \tag{LI}$$

called the left idempotency: indeed, for all x, y, there exists z such that $xz = y$ and now $x \cdot y = x \cdot (x \cdot z) = (x \cdot x) \cdot (x \cdot z) = (x \cdot x) \cdot y$. It would be therefore tempting to replace the idempotency in Larue's theorem by the left idempotency. Actually, T. Kepka [6] and P. Dehornoy [2] proved the following:

Theorem 2 (Dehornoy, Kepka) *The following varieties coincide:*

- *the variety generated by the left cancellative left distributive left idempotent (LCLDLI) groupoids;*
- *the variety generated by the left distributive left quasigroups (LDLQ);*
- *the variety generated by the groupoids of the half-conjugacy—given a group G and a subset X of G, the half-conjugacy is the operation $(a, x)\hat{\ }(b, y) = (axa^{-1}b, y)$, where $a, b \in G$ and $x, y \in X$.*

In order to have a complete analogy the theorem Joyce–Kepka–Larue, it remains to prove that the variety LDLD is the same as LDLQ = LCLDLI. One inclusion is trivial and the second one remains an open question. In this paper, we tackle the problem, showing a partial result.

If LDLD \ CLDLI happens to be nonempty, there must exist an identity that is satisfied in every LCLDLI groupoid but not in every LDLD one. The first choice is to look at some identities that hold in LDLQ and not in LD. Some of them were found by Larue [7] in the idempotent case. The shortest pair of terms that are equivalent in GC and not in LDI is

$$((a \cdot b) \cdot b)) \cdot (a \cdot c) \text{ and } (a \cdot b) \cdot ((b \cdot a) \cdot c). \tag{1}$$

It was however proved in [8] that these terms are equivalent in LDLD.

In [7], Larue actually presented an infinite family of identities that hold in GC and not in LDI. However, as the identities are constructed in a similar manner as (1), there is little hope that some of them is a counterexample to LDLD=LDLQ.

There is, actually, a new family of identities that hold in GC and do not hold in LDI: they were discovered by J. Barborikov [1] and, in fact, they form a broader family of identities that includes all Larue's ones. So far, we do not know, whether these identities bring anything new to our study of LDLI groupoids.

The aim of our article is different: to prove the hypothesis, not to reject it. We have a partial result only—we study the class of LDLD groupoids that satisfy a certain natural property, namely that the mapping $a \mapsto a^2$ is surjective. In Section 1 we present the mapping and we show some of its properties. In Section 2 we prove that all LDLD groupoids with $a \mapsto a^2$ surjective lie in the variety LDLQ.

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1 Squaring mapping

As we work with non-associative algebras, many parentheses are formally needed. Nevertheless, when working with LD groupoids, it is common to spare them. We write $xy \cdot z$ instead of $(x \cdot y) \cdot z$ and omitted parentheses mean branching to the right, i.e. $xyz = x \cdot yz$.

In this section we introduce a mapping called S_G (meaning squaring or successor) that plays a central rôle in our investigation. We start by recalling a structural property of LDLI groupoids.

Proposition 1 ([4]) *Let G be an LDLI groupoid. We define* ip_G *to be the smallest equivalence on G containing the pairs (a, a^2). Then*

- *For all a, b, c in G, if $(a, b) \in \mathrm{ip}_G$ then $ac = bc$.*
- ip_G *is a congruence of G with its classes being subgroupoids of G.*

Every class of ip_G is thus a subgroupoid of G satisfying the identity $yx = zx$; such a groupoid is essentially a unary algebra. It is natural to denote by S (successor) the derived unary operation on each of the class. Or more precisely, we define $S_G(x) = x \cdot x$ as a unary operation on G. Moreover, it is easy to see that S_G is an endomorphism of G, for any LDLI groupoid G.

Proposition 2 *Let G be an LDLI groupoid.*

(i) *If G is left cancellative then the endomorphism S_G is injective;*

(ii) *If G is a left quasigroup then the endomorphism S_G is bijective.*

Proof (i) If $a^2 = b^2$ then $(a, b) \in \mathrm{ip}_G$ and $a^2 = b^2 = ab$ results in $a = b$ due to the left cancellativity.

(ii) Take $a \in G$. The left divisibility guarantees the existence of an $x \in G$ satisfying $ax = a$. Now $a^2 = (ax)^2 = a \cdot x^2$. And the left cancellativity gives $a = x^2$. □

Under the impression of the previous proposition, it is natural to expect that the squaring is surjective in the case of LDLD groupoids. In fact, there exists neither a known counterexample nor a proof of the fact. Hence it is worth to take a closer look at S_G and try to find some equivalent translations.

Proposition 3 *Let G be an LDLI groupoid. The following conditions are equivalent:*

(i) *The endomorphism $a \mapsto a^2$ is onto.*

(ii) *For each a in G, there exists an element x in G, satisfying $a \cdot x = a$ and $(a, x) \in \mathrm{ip}_G$.*

(iii) *Every class of ip_G is a left divisible groupoid.*

Proof (i)⇒(ii): Given $a \in G$, there exists x satisfying $x^2 = a$. Now $a = x^2 = x^2 \cdot x = a \cdot x$.
(ii)⇒(i): Let x be an element satisfying $ax = a$ and $(a, x) \in \mathrm{ip}_G$. According to Proposition 1, we have $xx = ax$.
(ii)⇒(iii): Let b and c be ip_G-equivalent elements in G. We want to find an element x within the same congruence class, satisfying $bx = c$. But there exists x, satisfying $cx = c$ and $(c, x) \in \mathrm{ip}_G$. And, according to Proposition 1, we have $cx = bx$.
(iii)⇒(ii): Evident. □

The proposition tells us that the behavior of S_G in LDLD differs from the behavior of S_G in LCLDLI, although it looks similar on the first sight—we want a property to carry over to some subgroupoids. And subgroupoids of left divisible groupoids need not be left divisible in general.

2 Equality of varieties

In this section we prove that every LDLD groupoid with epimorphic S_G lies in the variety generated by LCLDLI. We will try to follow the same argumentation as Larue used when proving the similar theorem for LDI; we just replace every occurrence of the idempotency in his proof by the left idempotency. The proof however implicitly needs the fact that S_G is surjective—a fact that holds trivially in the idempotent case.

First we measure how far is an LD groupoid from being left cancellative.

Lemma 1 (Kepka [6]) *Let G be an LD groupoid. The relation $\sim$ defined by $a \sim b \Leftrightarrow x_1 x_2 \cdots x_n a = x_1 x_2 \cdots x_n b$, for some $x_1, \ldots, x_n$ in G, is the smallest congruence on G such that $G/_\sim$ is left cancellative.*

The following lemma is the key lemma. We prove that LDLD groupoids with surjective S_G satisfy all LCLDLI identities of a special form.

Lemma 2 *Let G be an LDLD groupoid with surjective S_G. For any variables $g_1, \ldots, g_m$ and terms u, v in n variables, the equality $g_m \cdots g_1 u \overset{LDLI}{=} g_m \cdots g_1 v$ implies that u and v have the same evaluation in G.*

Proof Suppose that we have

$$g_m \cdots g_1 u \stackrel{\text{LDLI}}{=} g_m \cdots g_1 v$$

for some variables $g_1, \ldots, g_m$ and terms u, v. The terms u and v are terms in variables $x_1, \ldots, x_n$, which can be written as $u(x_1, \ldots, x_n)$, respectively $v(x_1, \ldots, x_n)$. Let us take $a_1, \ldots, a_n$ in G arbitrary. We want to show

$$u(a_1, \ldots, a_n) = v(a_1, \ldots, a_n).$$

Each g_i can be written as some x_j. Denote $g_i = x_{\sigma(i)}$. We claim by induction that, for each $0 \leq i \leq m$, there exist $b_1, \ldots, b_n$ from G such that

$$u(a_1, \ldots, a_n) = b_{\sigma(i)} b_{\sigma(i-1)} \cdots b_{\sigma(1)} u(b_1, \ldots, b_n),$$
$$v(a_1, \ldots, a_n) = b_{\sigma(i)} b_{\sigma(i-1)} \cdots b_{\sigma(1)} v(b_1, \ldots, b_n).$$

For $i = 0$ we put $b_i = a_i$ and the result is vacuously true. Suppose now that all such b_k exist for some i and let us prove the result for $i + 1$. For each $1 \leq k \leq n$ we put b'_k to be an element satisfying $b_{\sigma(i+1)} b'_k = b_k$, such elements exist due to the left divisibility. Moreover, we want $(b_{\sigma(i+1)}, b'_{\sigma(i+1)}) \in \mathrm{ip}_G$, which is guaranteed by Proposition 3. Now

$$\begin{aligned} u(a_1, a_2, \ldots, a_n) &= b_{\sigma(i)} b_{\sigma(i-1)} \cdots b_{\sigma(1)} u(b_1, \ldots, b_n) \\ &= (b_{\sigma(i+1)} b'_{\sigma(i)}) \cdot (b_{\sigma(i+1)} b'_{\sigma(i-1)}) \cdots (b_{\sigma(i+1)} b'_{\sigma(1)}) \\ &\qquad \cdot u(b_{\sigma(i+1)} b'_1, \ldots, b_{\sigma(i+1)} b'_n) \\ &= b_{\sigma(i+1)} \cdot b'_{\sigma(i)} b'_{\sigma(i-1)} \cdots b'_{\sigma(1)} \cdot u(b'_1, \ldots, b'_n) \\ &= b'_{\sigma(i+1)} b'_{\sigma(i)} b'_{\sigma(i-1)} \cdots b'_{\sigma(1)} \cdot u(b'_1, \ldots, b'_n) \end{aligned}$$

and similarly for v, which finishes the induction.

Now,

$$u(a_1, \ldots, a_n) = b_{\sigma(m)} b_{\sigma(m-1)} \cdots b_{\sigma(1)} \cdot u(b_1, \ldots, b_n)$$
$$= (x_{\sigma(m)} x_{\sigma(m-1)} \cdots x_{\sigma(1)} \cdot u)(b_1, \ldots, b_n) = g_m g_{m-1} \cdots g_1 u(b_1, \ldots, b_n)$$

and similarly for v. Since

$$g_m g_{m-1} \cdots g_1 u \stackrel{\text{LDLI}}{=} g_m g_{m-1} \cdots g_1 v,$$

we get $u(a_1, \ldots, a_n) = v(a_1, \ldots, a_n)$ as desired. □

Proposition 4 (D. Larue [7]) *For any terms $w_1, \ldots, w_k$ there exist integers m, l, variables $g_1, \ldots, g_m$ and terms $p_1, \ldots, p_l$ such that*

$$g_m \cdots g_2 g_1 u \stackrel{LD}{=} p_l \cdots p_1 w_k \cdots w_1 u$$

for any term u.

Proposition 5 *Let G be an LDLD groupoid with S_G surjective. Then G lies in the variety generated by LCLDLI.*

Proof We prove alternatively that G satisfies any identity from the equational theory of LCLDLI. Consider arbitrary two terms u, v with $u \overset{\text{LCLDLI}}{=} v$. According to Lemma 1, there exist terms $w_1, \ldots, w_k$ such that $w_k \cdots w_1 u \overset{\text{LDLI}}{=} w_k \cdots w_1 v$. According to Proposition 4, there exist variables $g_1, \ldots, g_m$ and terms $p_1, \ldots, p_l$ such that $g_m \cdots g_1 z \overset{\text{LD}}{=} p_l \cdots p_1 w_k \cdots w_1 z$ for all z. Now

$$g_m \cdots g_1 u \overset{\text{LD}}{=} p_l \cdots p_1 w_k \cdots w_1 u \overset{\text{LDLI}}{=} p_l \cdots p_1 w_k \cdots w_1 v \overset{\text{LD}}{=} g_m \cdots g_1 v$$

and we apply Lemma 2.

Since G satisfies any identity from the equational theory of LCLDLI, it has to lie in the variety. □

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