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Convergence Results for Jungck-type Iterative Processes in Convex Metric Spaces

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Abstract

In this paper, the convergence results of [V. Berinde; A convergence theorem for Mann iteration in the class of Zamfirescu operators, Analele Universitatii de Vest, Timisoara, Seria Matematica-Informatica 45 (1) (2007), 33–41], [V. Berinde; On the convergence of Mann iteration for a class of quasi-contractive operators, Preprint, North University of Baia Mare (2003)] and [V. Berinde; On the Convergence of the Ishikawa Iteration in the Class of Quasi-contractive Operators, Acta Math. Univ. Comenianae 73 (1) (2004), 119–126] are extended from arbitrary Banach space setting to the convex metric space by weakening further the conditions on the parameter sequence $\{\alpha_n\} \subset [0,1]$. We establish the convergence of Jungck–Mann and Jungck–Ishikawa iterative processes for two nonselfmappings in a convex metric space setting by employing a general contractive condition. Similar results are also deduced for the Mann and Ishikawa iterations. Our results generalize, extend and improve a multitude of results in the literature including those of Berinde mentioned above.

Key words: arbitrary Banach space setting, Jungck–Mann and Jungck–Ishikawa iterative processes, convex metric space

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7 Introduction

The notion of convexity in metric spaces was introduced by Takahashi [29] and he established that all normed spaces and their convex subsets are convex metric spaces. In addition, he also gave several examples of convex metric spaces which are not imbedded in any normed space or Banach space. Several papers have been devoted to the study of convex metric spaces in the literature (see Agarwal et al [1], Beg [3, 4], Ciric [13], Guay et al [14] and Shimizu and Takahashi [27]).

Definition 7.1 [4, 29]: Let (X, d) be a metric space. A mapping $W\colon X \times X \times [0,1] \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x,\ y,\ \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y). \tag{1}$$

A metric space X having the convex structure W is called a *convex metric space*.

Let (X, d, W) be a convex metric space. A nonempty subset E of (X, d, W) is said to be *convex* if $W(x, y, \lambda) \in E$ whenever $(x, y, \lambda) \in E \times E \times [0,1]$.

Takahashi [29] has also shown that the open ball $B(x, r) = \{x \in X \mid d(x, y) < r\}$ and the closed ball $\overline{B(x, r)} = \{x \in X \mid d(x, y) \leq r\}$, are convex. See also [1, 4].

Zamfirescu [30] established a nice generalization of the Banach's fixed point theorem by employing the following contractive condition: For a mapping $T\colon E \to E$, there exist real numbers α, β, γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$, $0 \leq \gamma < \frac{1}{2}$ respectively such that for each $x, y \in E$, at least one of the following is true:

$$\left.\begin{array}{ll} (z_1) & d(Tx, Ty) \leq \alpha d(x, y) \\ (z_2) & d(Tx, Ty) \leq \beta\left[d(x, Tx) + d(y, Ty)\right] \\ (z_3) & d(Tx, Ty) \leq \gamma\left[d(x, Ty) + d(y, Tx)\right]. \end{array}\right\} \tag{2}$$

The mapping $T\colon E \to E$ satisfying (2) is called the *Zamfirescu contraction*. Any mapping satisfying condition (z_2) of (2) is called a *Kannan mapping*, while the mapping satisfying condition (z_3) is called *Chatterjea operator*. For more on conditions (z_2) and (z_3), we refer to Kannan [17] and Chatterjea [10] respectively. It has been shown in Berinde [5, 6, 8] that the contractive condition (2) implies

$$d(Tx, Ty) \leq 2\delta d(x, Tx) + \delta d(x, y), \quad \forall x, y \in E, \tag{3}$$

where $\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$, $0 \leq \delta < 1$.

Consequently, the author [5, 6, 8] used (3) to prove strong convergence results in Banach space setting for some iterative processes. More recently, Berinde [9] established several generalizations of Banach's fixed point theorem. In one of the results of [9], the following contractive condition was employed: For a mapping $T\colon E \to E$, there exists $\alpha \in [0,1)$ and some $L \geq 0$ such that for all $x, y \in E$, we have

$$d(Tx, Ty) \leq \alpha M_1(x, y) + L m(x, y), \tag{4}$$

where $M_1(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$, and $m(x, y) = \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

There are several iterative processes in the literature for which the fixed points of operators have been approximated over the years by various authors and we shall discuss some of these iterative processes in the present paper.

Singh et al [28] defined the following general iterative process to prove some stability results: Let $S, T\colon Y \to E$ and $T(Y) \subseteq S(Y)$. For any $x_0 \in Y$, define $\{Sx_n\}_{n=0}^{\infty} \subset E$ iteratively by

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots \tag{5}$$

A special case of (5) as shown in [28] is the following Jungck-type iteration: For any $x_0 \in Y$ and $T(Y) \subseteq S(Y)$,

$$Sx_{n+1} = f(T, x_n) = Tx_n, \quad n = 0, 1, 2, \ldots, \tag{6}$$

and also the following Jungck–Mann type iteration:

$$Sx_{n+1} = f(T, x_n) = (1 - \alpha_n)Sx_n + \alpha_n Tx_n, \quad \alpha_n \in [0, 1],\ n = 0, 1, 2, \ldots, \tag{7}$$

for any $x_0 \in Y$ and $T(Y) \subseteq S(Y)$.

Furthermore, if $Y = E$ and S =identity operator, then (7) reduces to the Mann iterative process

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \ldots,\ \alpha_n \in [0, 1]. \tag{8}$$

See [20] for Mann iterative process.

If $Y = E$, and $f(T, x_n) = Tx_n$, $n = 0, 1, 2 \ldots$, then (5) reduces to the Jungck iteration. See [16] for Jungck iteration. Jungck [16] established that the maps S and T satisfying

$$d(Tx, Ty) \leq ad(Sx, Sy), \quad \forall x, y \in E,\ a \in [0, 1), \tag{9}$$

have a unique common fixed point in complete metric space E, provided that S and T commute, $T(Y) \subseteq S(Y)$ and S is continuous.

Recently, in the paper Olatinwo [21], some stability and strong convergence results were proved in a Banach space setting for nonselfmappings using the following Jungck–Ishikawa iterative process: Let $(E, \|.\|)$ be a Banach space and Y an arbitrary set. Let $S, T\colon Y \to E$ be two nonselfmappings such that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of E and S is injective. Then, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty} \subset E$ iteratively by

$$\left.\begin{aligned} Sx_{n+1} &= (1 - \alpha_n)Sx_n + \alpha_n Tb_n \\ Sb_n &= (1 - \beta_n)Sx_n + \beta_n Tx_n \end{aligned}\right\}, \quad n = 0, 1, 2, \ldots, \tag{10}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$.

From (10), if $Y = E$ and S =identity operator, then we obtain Ishikawa iterative process (See [15]).

Remark 7.1 It has been shown in [21] that the iterative processes defined in (6), (7), (8) as well as those of Jungck [16] and some others in the literature are special cases of the iterative process defined in (10).

8 Preliminaries

Motivated by condition (4), we now state the following contractive condition which shall be used in proving our results:

Definition 8.1 Let (E, d, W) be a complete convex metric space and Y a nonempty closed convex subset of E. For two nonselfmappings $S, T\colon Y \to E$ with $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E, there exist $\delta \in [0, 1)$ and some $L \geq 0$ such that for all $x, y \in Y$, we have

$$d(Tx, Ty) \leq \delta d(Sx, Sy) + Lu(x, y), \tag{11}$$

where

$$u(x, y) = \min\{d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx), \\ \tfrac{1}{2}[d(Sx, Tx) + d(Sy, Ty)], \tfrac{1}{2}[d(Sx, Ty) + d(Sy, Tx)]\}.$$

Remark 8.1 (i) Condition (11) is independent of (4).

(ii) If in (11), E is a complete metric space with $E = Y, S =$ identity operator and $u(x, y) = d(x, Tx)$, then we obtain the contractive condition of Theorem 2.3 (Berinde [9]).

(iii) Condition (11) is reducible to those of Banach [2], Chatterjea [10], Jungck [16], Kannan [17], Popescu [22] and some others in the literature.

We shall express the iterative processes (7) and (10) in terms of convex structure in the following respectively: Let (E, d, W) be a complete convex metric space and Y a nonempty closed convex subset of E. Let $S, T\colon Y \to E$ be two nonselfmappings such that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of E and S is injective.

Then, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty} \subset E$ iteratively by

$$Sx_{n+1} = W(Sx_n, Tx_n, \alpha_n), \quad \alpha_n \in [0, 1]. \tag{12}$$

Also, for $x_0 \in Y$, define the sequence $\{Sx_n\}_{n=0}^{\infty} \subset E$ iteratively by

$$Sx_{n+1} = W(Sx_n, Tb_n, \alpha_n), \quad Sb_n = W(Sx_n, Tx_n, \beta_n), \quad \alpha_n, \beta_n \in [0, 1]. \tag{13}$$

Remark 8.2 The iterative process (12) is the form of (7) (that is, Jungck–Mann iterative process) in convex metric space setting while (13) is a similar form of (10) (that is, Jungck–Ishikawa iterative process) in convex metric space setting.

Definition 8.2 [25, 26]: Let X and Y be two nonempty sets and $S, T\colon X \to Y$ two mappings. Then, an element $x^* \in X$ is a *coincidence point* of S and T if and only if $Sx^* = Tx^*$.

Denote the set of the coincidence points of S and T by $C(S, T)$ and when $C(S, T)$ is nonempty, we denote it by $C(S, T) \neq \phi$.

9 Main Results

Theorem 9.1 *Let (E, d, W) be a complete convex metric space and Y a nonempty closed convex subset of E. Suppose that $S, T\colon Y \to E$ are nonselfmappings satisfying condition (11) such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E and S is an injective mapping. Let $C(S,T)$ be the set of coincidence points of S and T, with $C(S,T) \neq \phi$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ defined by (12) be the Jungck–Mann iterative process with $\alpha_n \in [0,1]$, such that $0 < \alpha \leq \alpha_n$, $\forall n$. Then, the Jungck–Mann iteration $\{Sx_n\}_{n=0}^{\infty}$ converges to p.*

Proof Let $C(S,T)$ be the set of the coincidence points of S and T. We employ condition (11) to establish that S and T have a unique coincidence point z (i.e. $Sz = Tz = p$ (say)): Injectivity of S is sufficient. Suppose that there exist $z_1, z_2 \in C(S,T)$ such that $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$.

If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$.

If $p_1 \neq p_2$, then we have by the contractiveness condition (11) for S and T that

$$\begin{aligned}
&0 < ||p_1 - p_2|| = ||Tz_1 - Tz_2|| \\
&\leq \delta d(Sz_1, Sz_2) + L\min\{d(Sz_1,Tz_1), d(Sz_2,Tz_2), d(Sz_1,Tz_2), d(Sz_2,Tz_1), \\
&\qquad \tfrac{1}{2}[d(Sz_1,Tz_1) + d(Sz_2,Tz_2)], \tfrac{1}{2}[d(Sz_1,Tz_2) + d(Sz_2,Tz_1)]\} \\
&\qquad\qquad = \delta d(p_1,p_2) + \min\{0, d(p_1,p_2)\} = \delta d(p_1,p_2),
\end{aligned}$$

from which it follows that $1 - \delta > 0$ since $\delta \in [0,1)$, but $d(p_1,p_2) \leq 0$, which is a contradiction since metric is nonnegative. Therefore, we have that $d(p_1,p_2) = 0$, that is, $p_1 = p_2 = p$. Since $p_1 = p_2$, then we have that $p_1 = Sz_1 = Tz_1 = Sz_2 = Tz_2 = p_2$, leading to $Sz_1 = Sz_2 \Longrightarrow z_1 = z_2 = z$ (since S is injective). Hence, $C(S,T) = \{z\}$, that is, z is a unique coincidence point of S and T.

We now establish that $\{Sx_n\}_{n=0}^{\infty}$ converges to p (where $Sz = Tz = p$) using again, condition (11). Therefore, we have by (12) that

$$\begin{aligned}
d(Sx_{n+1}, p) &= d(W(Sx_n, Tx_n, \alpha_n), p) \\
&\leq (1-\alpha_n) d(Sx_n, p) + \alpha_n d(Tx_n, p) \\
&= (1-\alpha_n) d(Sx_n, p) + \alpha_n d(Tz, Tx_n). \qquad (14)
\end{aligned}$$

We have by (11) that

$$\begin{aligned}
d(Tz, Tx_n) &\leq \delta d(Sz, Sx_n) \\
&\quad + L\min\{d(Sz,Tz), d(Sx_n,Tx_n), d(Sz,Tx_n), d(Sx_n,Tz), \\
&\quad \tfrac{1}{2}[d(Sz,Tz) + d(Sx_n,Tx_n)], \tfrac{1}{2}[d(Sz,Tx_n) + d(Sx_n,Tz)]\} \\
&= \delta d(p, Sx_n). \qquad (15)
\end{aligned}$$

Using (15) in (14) yields

$$\begin{aligned} d(Sx_{n+1},p) &\le [1-(1-\delta)\alpha_n]d(Sx_n,p) \\ &\le [1-(1-\delta)\alpha]d(Sx_n,p) \\ &\le [1-(1-\delta)\alpha]^2 d(Sx_{n-1},p) \le \cdots \le [1-(1-\delta)\alpha]^{n+1}d(Sx_0,p) \\ &\to 0 \text{ as n} \to \infty, \text{ since } 0 < 1-(1-\delta)\alpha < 1. \end{aligned}$$

Hence, we obtain that $d(Sx_{n+1},p) \to 0$ as $n \to \infty$, that is, $\{Sx_n\}_{n=0}^{\infty}$ converges to p. □

Theorem 9.2 *Let (E,d,W) be a complete convex metric space and Y a nonempty closed convex subset of E. Suppose that $S,T\colon Y \to E$ are nonselfmappings satisfying condition (11) such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E and S is an injective mapping. Let $C(S,T)$ be the set of coincidence points of S and T, with $C(S,T) \neq \phi$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ defined by (13) be the Jungck–Ishikawa iterative process with $\alpha_n, \beta_n \in [0,1]$, such that $0 < \alpha \le \alpha_n$, $0 < \beta \le \beta_n$, $\forall n$. Then, the Jungck–Ishikawa iteration $\{Sx_n\}_{n=0}^{\infty}$ converges to p.*

Proof The proof of the uniqueness of the coincidence point of S and T has been established in Theorem 3.1 by using condition (11).

We now prove that $\{Sx_n\}_{n=0}^{\infty}$ converges to p (where $Sz = Tz = p$) using again, condition (11). Therefore, we have (13) that

$$\begin{aligned} d(Sx_{n+1},p) &= d(W(Sx_n,Tb_n,\alpha_n),p) \\ &\le (1-\alpha_n)d(Sx_n,p) + \alpha_n d(Tb_n,p) \\ &= (1-\alpha_n)d(Sx_n,p) + \alpha_n d(Tz,Tb_n). \end{aligned} \tag{16}$$

Also, we have by (11) again that

$$\begin{aligned} d(Tz,Tb_n) &\le \delta d(Sz,Sb_n) \\ &\quad + L\min\{d(Sz,Tz), d(Sb_n,Tb_n), d(Sz,Tb_n), d(Sb_n,Tz), \\ &\quad \tfrac{1}{2}[d(Sz,Tz)+d(Sb_n,Tb_n)], \tfrac{1}{2}[d(Sz,Tb_n)+d(Sb_n,Tz)]\} \\ &= \delta d(Sz,Sb_n). \end{aligned} \tag{17}$$

Using (17) in (16) gives

$$d(Sx_{n+1},p) \le (1-\alpha_n)d(Sx_n,p) + \alpha_n\delta d(Sz,Sb_n). \tag{18}$$

Using (13) and (15) gives

$$\begin{aligned} d(Sz,Sb_n) &= d(Sz,W(Sx_n,Tx_n,\beta_n)) \\ &\le (1-\beta_n)d(Sz,Sx_n) + \beta_n d(Sz,Tx_n) \\ &= (1-\beta_n)d(p,Sx_n) + \beta_n d(Tz,Tx_n) \\ &\le [1-(1-\delta)\beta_n]d(Sx_n,p). \end{aligned} \tag{19}$$

Using (19) in (18) yields

$$\begin{aligned} d(Sx_{n+1},p) &\le [1-(1-\delta)\alpha_n-(1-\delta)\delta\alpha_n\beta_n)]d(Sx_n,p)\\ &\le [1-(1-\delta)\alpha-(1-\delta)\delta\alpha\beta)]d(Sx_n,p)=\gamma d(Sx_n,p)\\ &\le \gamma^2 d(Sx_{n-1},p)\le\cdots\le\gamma^{n+1}d(Sx_0,p)\to 0 \text{ as n}\to\infty, \end{aligned}$$

since $0<\gamma=1-(1-\delta)\alpha-(1-\delta)\delta\alpha\beta)<1$. Hence, we obtain again as in Theorem 3.1 that $d(Sx_{n+1},p)\to 0$ as $n\to\infty$, that is, $\{Sx_n\}_{n=0}^{\infty}$ converges to p. □

Theorem 9.3 *Let (E,d,W) be a complete convex metric space and Y a nonempty closed convex subset of E. Suppose that $T\colon Y\to Y$ is a selfmapping satisfying the condition*

$$d(Tx,Ty)\le\delta d(x,y)+Lv(x,y),\quad \forall x,y\in Y,\ L\ge 0,\ \delta\in[0,1),$$

where

$$\begin{aligned} v(x,y)=\min\{&d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx),\\ &\tfrac12[d(x,Tx)+d(y,Ty)],\tfrac12[d(x,Ty)+d(y,Tx)]\}. \end{aligned}$$

Let p be a fixed point of T. For $x_0\in Y$, let $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1}=W(x_n,Tx_n,\alpha_n),$$

be the Mann iterative process with $\alpha_n\in[0,1]$, such that $0<\alpha\le\alpha_n$, $\forall n$. Then, the Mann iteration $\{x_n\}_{n=0}^{\infty}$ converges to p.

Theorem 9.4 *Let (E,d,W) be a complete convex metric space and Y a nonempty closed convex subset of E. Suppose that $T\colon Y\to Y$ is a selfmapping satisfying the condition*

$$d(Tx,Ty)\le\delta d(x,y)+Lv(x,y),\quad \forall x,y\in Y,\ L\ge 0,\ \delta\in[0,1),$$

where

$$\begin{aligned} v(x,y)=\min\{&d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx),\\ &\tfrac12[d(x,Tx)+d(y,Ty)],\tfrac12[d(x,Ty)+d(y,Tx)]\}. \end{aligned}$$

Let p be a fixed point of T. For $x_0\in Y$, let $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1}=W(x_n,Tb_n,\alpha_n),\qquad b_n=W(x_n,Tx_n,\beta_n),$$

be the Ishikawa iterative process with α_n, $\beta_n\in[0,1]$ such that $0<\alpha\le\alpha_n$, $0<\beta\le\beta_n$, $\forall n$. Then, the Ishikawa iteration $\{x_n\}_{n=0}^{\infty}$ converges to p.

Remark 9.1 Theorem 3.1–Theorem 3.4 generalize, extend and improve a multitude of results. In particular, Theorem 3.2 and Theorem 3.4 are generalizations and extensions of the results of Berinde [5, 6, 8], Theorem 2 and Theorem 3 of Kannan [18], Theorem 3 of Kannan [19], Theorem 4 of Rhoades [23] as well as Theorem 8 of Rhoades [24]. Also, both Theorem 4 of Rhoades [23] and Theorem 8 of Rhoades [24] are Theorem 4.9 and Theorem 5.6 of Berinde [7] respectively. Theorem 3.1 and Theorem 3.3 also generalize and extend the result of Berinde [5, 8], both Theorem 2 and Theorem 3 of Kannan [18], Theorem 3 of Kannan [19] as well as Theorem 4 of Rhoades [23]. See also some similar results of the author [21].

Remark 9.2 In the results of [5, 6, 8, 21], the condition on $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ is $\sum_{n=0}^{\infty} \alpha_n = \infty$, and this has now been removed and replaced by the conditions: $0 < \alpha \leq \alpha_n$, $0 < \beta \leq \beta_n$. Therefore, our results are improvements over the previous ones in the literature.

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