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Pseudocomplemented and Stone Posets*

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Abstract

We show that every pseudocomplemented poset can be equivalently expressed as a certain algebra where the operation of pseudocomplementation can be characterized by means of remaining two operations which are binary and nullary. Similar characterization is presented for Stone posets.

Key words: pseudocomplement, pseudocomplemented poset, Stone poset

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The concept of pseudocomplement was introduced by O. Frink [2] for meet-semilattices, Stone lattices were studied by R. Balbes and A. Horn [1]. S. K. Nimbhokar and A. Rahemani [3] modified the approach developed for posets by P. V. Venkatarasimhan [4] and use it for characterization of Stone join-semilattices.

The aim of this paper is to get another approach which goes in a sense conversely. We will show that every pseudocomplemented poset can be organized in a certain algebra. This can be analogously done for Stone posets.

Let us recall that the concept of pseudocomplement in a poset with the least element 0 was introduced in [4] by means of order-ideals. However, it can be easily paraphrased as follows.

Definition 1 Let $\mathcal{P} = (P; \leq, 0)$ be a poset with the least element 0, let $a \in P$. We say that $a^* \in P$ is a *pseudocomplement* of a if

(i) there exists the infimum $a \wedge a^*$ of $\{a, a^*\}$ and is equal to 0;

(ii) if $b \in P$ and $a \wedge b$ exists and equals 0, then $b \leq a^*$.

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A poset $\mathcal{P} = (P; \leq, 0)$ is called *pseudocomplemented* if there exists a pseudocomplement a^* for each $a \in P$. This fact will be expressed by notation $\mathcal{P} = (P; \leq, 0, ^*)$.

Convention In what follows, the notation $a \wedge b = c$ will be read as "the infimum $a \wedge b$ exists and is equal to c".

Example 1 Consider the poset $\mathcal{P} = (\{0, a, b, c, d, 1\}; \leq, 0)$ vizualized in Fig. 1:

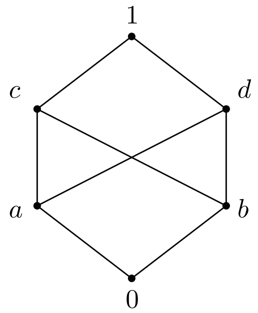

Fig. 1

Evidently, $\mathcal{P}$ is neither a lattice nor a meet-semilattice. However, $\mathcal{P}$ is pseudocomplemented and the pseudocomplements are determined by Definition 1 as follows

x	0	a	b	c	d	1
x^*	1	b	a	0	0	0

.

The following is a trivial consequence of the definition.

Lemma 1 *Let $\mathcal{P} = (P; \leq, 0)$ be a pseudocomplemented poset. Then*

(a) *$\mathcal{P}$ has the greatest element $1 = 0^*$;*

(b) *$x \leq x^{**}$, $x^{***} = x^*$ and if $x \leq y$, then $y^* \leq x^*$, for all $x, y \in P$.*

We show now that a certain algebra of type $(2, 0)$ can be assigned to every poset $\mathcal{P} = (P; \leq, 0)$.

Definition 2 Let $\mathcal{P} = (P; \leq, 0)$ be a poset with the least element 0. Define a binary operation $\sqcap$ on $\mathcal{P}$ as follows: if $x \wedge y$ exists, then $x \sqcap y = x \wedge y$, and $x \sqcap y = 0$ otherwise. The algebra $\mathcal{A}(P) = (P; \sqcap, 0)$ will be called a *$\mathcal{P}$-algebra.*

Example 2 Consider the poset $\mathcal{P} = (\{0, a, b, c, d, 1\}; \leq, 0)$ of Example 1 (vizualized in Fig. 1). Then the corresponding $\mathcal{P}$-algebra $\mathcal{A}(P) = (\{0, a, b, c, d, 1\}; \sqcap, 0)$ is defined uniquelly by the operation table

$\sqcap$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	0	c
d	0	a	b	0	d	d
1	0	a	b	c	d	1

.

Remark 1 (a) It is obvious that the operation $\sqcap$ is commutative, i.e. $x \sqcap y = y \sqcap x$ for all $x, y \in P$.

(b) If $x \leq y$ then $x \wedge y$ exists and $x \wedge y = x$, i.e. also $x \sqcap y = x$. Conversely, if $x \sqcap y = x$ then either $x \wedge y$ exists, i.e. $x \wedge y = x$ and hence $x \leq y$, or $x \wedge y$ does not exist, i.e. $0 = x \sqcap y = x$ whence $x = 0 \leq y$ again. Thus we have

$$x \leq y \quad \text{if and only if} \quad x \sqcap y = x$$

in every $\mathcal{P}$-algebra $\mathcal{A}(P) = (P; \sqcap, 0)$.

Now, we prove that also conversely, every poset $\mathcal{P} = (P; \leq, 0)$ can be derived from its assigned $\mathcal{P}$-algebra $\mathcal{A}(P)$. For this, we characterize the operation $\sqcap$ of $\mathcal{A}(P)$ by several simple axioms.

Lemma 2 *Let $\mathcal{P} = (P; \leq, 0)$ be a poset with 0 and $\mathcal{A}(P) = (P; \sqcap, 0)$ the corresponding $\mathcal{P}$-algebra. Then the operations $\sqcap$ and 0 satisfy the following conditions:*

(A0) $x \sqcap 0 = 0$

(A1) $x \sqcap x = x$

(A2) $x \sqcap y = y \sqcap x$

(A3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$

(A4) *if there exists an element t such that (a) $x \sqcap t = t = y \sqcap t$ and (b) for all w, $x \sqcap w = w = y \sqcap w$ implies $w \sqcap t = w$, then $x \sqcap y = t$, and if such an element does not exist, then $x \sqcap y = 0$.*

Proof By Remark 1 we have $x \leq y$ iff $x \sqcap y = x$. Since 0 is the least element of $\mathcal{P}$, we have $x \sqcap 0 = 0$ which is (A0). The conditions (A1), (A2) follow directly by Definition 2. Further, $x \sqcap y \leq x$ and $(x \sqcap y) \sqcap z \leq x$, thus $x \sqcap ((x \sqcap y) \sqcap z) = x \wedge ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$ which is (A3). For (A4), assume that such an element t exists in $\mathcal{P}$. Then, by (a), $t \leq x$, $t \leq y$ and, by (b), it is the greatest element in P of this property, i.e. $t = x \wedge y$ and hence $x \sqcap y = t$. If it does not exist, then $x \sqcap y = 0$, proving (A4). □

Lemma 3 *Let $\mathcal{A} = (A; \sqcap, 0)$ be an algebra of type $(2, 0)$ satisfying (A0)–(A4). Define $x \leq y$ if $x \sqcap y = x$. Then $\mathcal{P}(A) = (A; \leq, 0)$ is a poset with the least element 0 and $x \sqcap y = x \wedge y$ provided $x \wedge y$ exists, and $x \sqcap y = 0$ otherwise.*

Proof By (A0) and (A2) we have $0 \leq x$ for each $x \in A$. By (A1) we obtain $x \leq x$, reflexivity of $\leq$. Assume $x \leq y$ and $y \leq x$. Then, by (A2), $x = x \sqcap y = y \sqcap x = y$ proving antisymmetry of $\leq$. If $x \leq y$ and $y \leq z$, i.e. $x \sqcap y = x$ and $y \sqcap z = y$, then by (A2) and (A3) we derive $x \sqcap z = (x \sqcap y) \sqcap z = (x \sqcap (y \sqcap z)) \sqcap z = x \sqcap (y \sqcap z) = x \sqcap y = x$ whence $\leq$ is also transitive, i.e. it is a partial order on A, thus $(A; \leq, 0)$ is a poset with the least element 0.

Assume now that $a, b \in A$ and $a \wedge b$ exists (with respect to the aforementioned order $\leq$). Then for $t = a \wedge b$ the assumptions of (A4) are satisfied and hence

$a \sqcap b = t = a \wedge b$. If $a \wedge b$ does not exist, then there is no $t \in A$ satisfying the assumptions of (A4) and hence $a \sqcap b = 0$. □

Let $\mathcal{A} = (A; \sqcap, 0)$ be an algebra satisfying (A0)–(A4). The poset $\mathcal{P}(A) = (A; \leq, 0)$ derived in Lemma 3 will be called the *induced poset*. We are going to show that posets $\mathcal{P}$ with 0 and the corresponding $\mathcal{P}$-algebras are in a one-to-one correspondence.

Lemma 4 *Let $\mathcal{P} = (P; \leq, 0)$ be a poset with 0, $\mathcal{A}(P) = (P; \sqcap, 0)$ the $\mathcal{P}$-algebra and $\mathcal{P}(\mathcal{A}(P)) = (P; \sqsubseteq, 0)$ the induced poset. Then $\mathcal{P} = \mathcal{P}(\mathcal{A}(P))$.*

Let $\mathcal{A} = (A; \sqcap, 0)$ be an algebra satisfying (A0)–(A4), $\mathcal{P}(A) = (A; \leq, 0)$ the induced poset and $\mathcal{A}(\mathcal{P}(A)) = (A; \cap, 0)$ its $\mathcal{P}(A)$-algebra. Then $\mathcal{A} = \mathcal{A}(\mathcal{P}(A))$.

Proof (a) We need to show $\leq = \sqsubseteq$. Assume $x \leq y$ in $\mathcal{P}$. By Remark 1, this is equivalent to $x \sqcap y = x$ in $\mathcal{A}(P)$ which is equivalent by definition to $x \sqsubseteq y$. Hence $\mathcal{P} = \mathcal{P}(\mathcal{A}(P))$.

(b) Assume $a \wedge b$ exists in $\mathcal{P}(A)$. Then $a \cap b = a \wedge b$ in $\mathcal{A}(\mathcal{P}(A))$ but also $a \sqcap b = a \wedge b$ in $\mathcal{A}$ by Lemma 3. In both cases, we obtain $a \cap b = a \sqcap b$ and hence $\mathcal{A} = \mathcal{A}(\mathcal{P}(A))$. □

Now, we are ready to characterize pseudocomplementation in posets by means of the corresponding $\mathcal{P}$-algebra.

Theorem 1 *Let $\mathcal{P} = (P; \leq, 0)$ be a poset with the least element 0, let $\mathcal{A}(P) = (P; \sqcap, 0)$ be its $\mathcal{P}$-algebra. Let * be a unary operation on P. Then $\mathcal{P} = (P; \leq, 0, ^*)$ is a pseudocomplemented poset if and only if $(P; \sqcap, ^*, 0)$ satisfies the following conditions:*

(P1) $x \sqcap 0^* = x$

(P2) $x \sqcap (x^* \sqcap y) = 0$

(P3) *if $x \sqcap (y \sqcap z) = 0$ for all $z \in P$, then $y \sqcap x^* = y$*

Proof Assume that $\mathcal{P} = (P; \leq, 0, ^*)$ is a pseudocomplemented poset. Then for each $x, y \in P$ we have $x^* \sqcap y \leq x^*$. Since $x \wedge x^*$ exists and is equal to 0, we conclude that also $x \wedge (x^* \sqcap y)$ exists and is equal to 0, i.e. $x \sqcap (x^* \sqcap y) = x \wedge (x^* \sqcap y)$ proving (P2). Assume $x \sqcap (y \sqcap z) = 0$ for each $z \in P$. If there exists $c \in P$ such that $c \neq 0$ and $x \sqcap y = c$ then, by (A2), (A3) and the assumption, $0 = x \sqcap (y \sqcap c) = x \sqcap (y \sqcap (x \sqcap y)) = y \sqcap (x \sqcap y) = x \sqcap y = c \neq 0$, a contradiction. Therefore $x \wedge y = 0$ whence $y \leq x^*$ and $y \sqcap x^* = y$ proving (P3). The condition (P1) is evident.

Conversely, let $(P; \sqcap, ^*, 0)$ satisfy (P1), (P2) and (P3). By (P1), 0^* is the greatest element of $\mathcal{P}$. If $y \leq x$ and $y \leq x^*$ then, according to (P2), we obtain $y = x \sqcap y = x \sqcap (x^* \sqcap y) = 0$. Hence $x \wedge x^* = 0$. Assume now $x \wedge z = 0$. Then $x \sqcap (z \sqcap c) \leq x$ and $x \sqcap (z \sqcap c) \leq z \sqcap c \leq z$ for each c, thus $x \sqcap (z \sqcap c) \leq x \wedge z = 0$. By (P3) we conclude $z \leq x^*$, i.e. x^* is the greatest element of P satisfying $x \wedge z = 0$, i.e. it is the pseudocomplement of x. □

We focus our attention on Stone posets in the rest of the paper. As in the previous case, the definition of [4] can be paraphrased as follows.

Definition 3 Let $\mathcal{P} = (P; \leq, 0, ^*)$ be a pseudocomplemented poset. Then $\mathcal{P}$ is called a *Stone poset* if for each $x \in P$ the supremum $x^* \vee x^{**}$ exists and equals 1 (where $1 = 0^*$).

Example 3 The poset from Example 1 is pseudocomplemented, but it is not a Stone one because, e.g., $a^* \vee a^{**} = b \vee a$ does not exist.

Example 4 Consider the poset $\mathcal{P} = (\{0, a, b, c, d, p, q, 1\}; \leq, 0)$ depicted in Fig. 2.

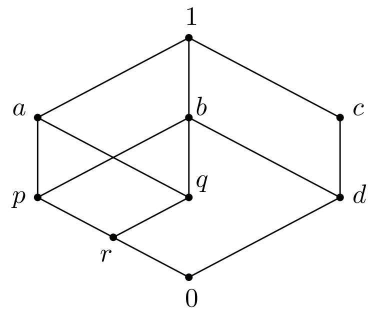

Fig. 2

Then $\mathcal{P}$ is pseudocomplemented, pseudocomplements are given by the table:

x	0	p	q	d	a	b	c	r	1
x^*	1	c	c	a	c	0	a	c	0
x^{**}	0	a	a	c	a	1	c	a	1

.

Since $a \vee c = 1$ and $0 \vee 1 = 1$, we have $x^* \vee x^{**} = 1$ for each $x \in P$, thus $\mathcal{P}$ is a Stone poset.

We proceed analogously as in the previous case. Consider a bounded poset $\mathcal{P} = (P; \leq, 0, 1)$. The operation $\sqcap$ on P is defined by Definition 2. Now we define $\sqcup$ on P dually: if $x \vee y$ exists, then $x \sqcup y = x \vee y$, and $x \sqcup y = 1$ otherwise. The algebra $\mathcal{B}(P) = (P; \sqcup, \sqcap, 0, 1)$ will be called the $\mathcal{P}_1$-*algebra* assigned to $\mathcal{P}$.

Remark 2 Analogously as in the previous case, one can easily check that $\sqcup$ has the properties:

(B0) $x \sqcup 1 = 1$

(B1) $x \sqcup x = x$

(B2) $x \sqcup y = y \sqcup x$

(B3) $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$

(B4) if there exists an element s such that (a) $x \sqcup s = s = y \sqcup s$ and (b) for all u, $x \sqcup u = u = y \sqcup u$ implies $u \sqcup s = u$, then $x \sqcup y = s$, and if such an element does not exist, then $x \sqcup y = 1$.

Moreover, $x \leq y$ iff $x \sqcup y = y$ and the operations $\sqcup$ and $\sqcap$ are connected via the absorption laws

$$x \sqcup (x \sqcap y) = x \quad \text{and} \quad x \sqcap (x \sqcup y) = x.$$

We will not repeat the analogues of Lemmas 1–4 for $\mathcal{P}_1$-algebras, the reader can prove them dually. With these concepts in hands, we can characterize Stone posets in terms of $\mathcal{P}_1$-algebras as follows.

Theorem 2 *Let $\mathcal{P} = (P; \leq, 0)$ be a poset with the least element 0. Let * be a unary operation on P. Then $\mathcal{P} = (P; \leq, 0, ^*)$ is a Stone poset if and only if it satisfies* (P1)–(P3) *and*

(P4) $x^* \sqcup (x^{**} \sqcup z) = 0^*$,

where $\sqcup$ and $\sqcap$ are operations of the corresponding $\mathcal{P}_1$-algebra $\mathcal{B}(P)$.

Proof Assume that $\mathcal{P} = (P; \leq, 0, ^*)$ is a Stone poset, $1 = 0^*$. Then it satisfies (P1)–(P3) by Theorem 1. Moreover, for each $x \in P$ there exists $x^* \vee x^{**}$ and is equal to 1. Since $x^{**} \leq x^{**} \sqcup z$, also $x^* \vee (x^{**} \sqcup z)$ must exist and hence $1 = x^* \vee (x^{**} \sqcup z) \leq x^* \sqcup (x^{**} \sqcup z)$ proving (P4).

Conversely, assume that the corresponding $\mathcal{P}_1$-algebra $\mathcal{B}(P)$ satisfies (P1)–(P4). By Theorem 1 and Lemma 1, $(P; \leq, 0, ^*)$ is a pseudocomplemented poset with the greatest element $1 = 0^*$. Let $x \in P$. Suppose $x^* \leq a$, $x^{**} \leq a$ for some $a \in P$. Then $x^* \sqcup a = a = x^{**} \sqcup a$ and hence $1 = x^* \sqcup (x^{**} \sqcup a) = x^* \sqcup a = a$. This means that $x^* \vee x^{**}$ exists and is equal to 1, i.e. $\mathcal{P}$ is a Stone poset. □

Remark 3 The reader acquainted with [3] can check that the poset in Fig. 2 is not modular. Hence, our characterization given in Theorem 2 works also in this case.

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