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Rank tests of symmetry and R-estimation of location parameter under measurement errors

Radim NAVRÁTIL

Department of Probability and Mathematical Statistics
Faculty of Mathematics and Physics, Charles University in Prague
Sokolovská 83, Praha 8, 18675, Czech Republic
e-mail: navratil@karlin.mff.cuni.cz
and
A. K. Md. Ehsanes Saleh, School of Mathematics and Statistics
Carleton University
1125 Colonel By Drive, Ottawa, Canada
e-mail: esaleh@math.carleton.ca

Dedicated to Lubomír Kubáček on the occasion of his 80th birthday

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Abstract

This paper deals with the hypotheses of symmetry of distributions with respect to a location parameter when the response variables are subject to measurement errors. Rank tests of hypotheses about the location parameter and the related R-estimators are studied in an asymptotic set up. It is shown, when and under what conditions, these rank tests and R-estimators can be used effectively, and the effect of measurement errors on the power of the test and on the efficiency of the R-estimators is indicated.

Key words: location, measurement error, R-estimate, rank, rank test

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1 Introduction

In practice we often need to test the hypothesis that a new treatment is better than the current, or that older twin has different properties than younger, or the left eye can see sharper than the right one. In all these situations we use one-sample test of symmetry. The basic idea of this test is that we observe the difference of two treatments, between younger and older twins, between left and right eye and suppose that its distribution is symmetric around Δ and test whether $\Delta = 0$. We may also be interested in the point and interval estimation of the location parameter Δ.

If we know that the distribution of the differences is normal, then it is logical that, for the test of hypotheses, we use the classical paired t-test, and for the point and interval estimator of the location parameter, we use the sample mean and student's t-confidence interval respectively. But often, this condition is not satisfied because of the nature of experiment and possible measurement errors that creep into the measuring instruments. For this situation we take recourse to nonparametric methods, which are the subject matter of this paper and we consider rank tests and rank estimators. One main feature of nonparametric methods is that it needs only a weak set of assumptions for its validity together with the robustness and high efficiency characteristics, relative to the parametric methods.

In many practical situations, the response variable of interest may not be obtainable accurately, instead they are effected by additive measurement errors. In this situation, parametric methods may not be suitable due to the absence of the knowledge of exact distributions of the measurement errors except for the restrictive assumption of normality of distributions. In this case, the rank tests and estimation may lead us to easy, simple and accessible solution to this problem. The immense growth of parametric methods during the last century left a vacuum for the nonparametric methods for the measurement error models so described until recently. The first set of articles on rank tests on regression model with measurement errors is due to Jurečková et al. [1], [2], [4] and Navrátil [8] and first set of articles on R-estimation is due to Saleh et al. [9], [10].

2 Rank tests of symmetry

Let $X_1, \dots, X_n$ be independent identically distributed (i.i.d.) random variables with an unknown continuous distribution function $F(x - \Delta)$ and continuous density $f(x - \Delta)$, where f is symmetric around zero and has finite Fisher information $I(f)$ (these conditions will be assumed in the whole article). We want to test the hypothesis $\mathbf{H}_0\colon \Delta = 0$ against the alternative $\Delta > 0$. The following theoretical results are summarizing existing results about this problem; hence the respective proofs are omitted, inquisitive reader may find them in books [6], [7], where there are described in detail with some additional comments. The locally most powerful rank test of hypothesis $\mathbf{H}_0$ has the critical region (see [6, p. 74])

$$\sum_{i=1}^{n} \tilde{a}_n^+(R_i^+, f)\,\mathrm{sign}(X_i) \geq k_\alpha,$$

where k_α is determined so that the significance level of the test is α and R_i^+ are the ranks of $|X_i|$ among $|X_1|, \dots, |X_n|$ and the scores $\tilde{a}_n^+(i, f)$ have form

$$\tilde{a}_n^+(i, f) = \mathbb{E}\tilde{\varphi}^+(U_{(i)}, f), \quad \tilde{\varphi}^+(u, f) = \tilde{\varphi}\left(\frac{u+1}{2}, f\right), \quad 0 < u < 1,$$

where

$$\tilde{\varphi}(u, f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))}, \quad 0 < u < 1$$

and $U_{(1)} \leq \ldots \leq U_{(n)}$ are the order statistics of a set of independent r.v. $U_1, \ldots, U_n$, each uniformly distributed over interval (0,1). The problem is, that we do not know the distribution of X_i – that is the reason why we use the rank tests. The other difficulty is to compute exact value of the scores $\tilde{a}_n^+(i, f)$. Hence we use the approximate scores $a_n^+(i) = \varphi^+\left(\frac{i}{n+1}\right)$, where $\varphi^+(u) = \varphi\left(\frac{u+1}{2}\right)$ and φ is a nondecreasing, square integrable function on $(0, 1)$ and finally denote

$$S_n = \sum_{i=1}^{n} a_n^+(R_i^+) \operatorname{sign}(X_i).$$

The exact distribution under null hypothesis is distribution-free and it is derived in detail in the book [7, p. 124–125], the asymptotic normality of S_n is proven in [6, p. 197–199], more precisely: Under the assumptions mentioned above and under $\mathbf{H}_0$ statistic S_n is asymptotically normal with 0 mean and variance $\sigma_n^2 = nA^2(\varphi^+)$, where $A^2(\varphi^+) = \int_0^1 [\varphi^+(u)]^2 du$, while under the local alternative $\mathbf{K}_n \colon \Delta = \frac{\Delta^*}{\sqrt{n}}$, where $\Delta^* \neq 0$ is fixed, S_n is asymptotically normal with mean μ_n and variance σ_n^2, where

$$\mu_n = \sqrt{n}\Delta^* \gamma(\varphi^+, f), \quad \text{with } \gamma(\varphi^+, f) = \int_0^1 \varphi^+(u)\tilde{\varphi}^+(u, f) du.$$

As being mentioned in the Introduction this procedure is mainly used for testing homogeneity in two populations. For example, we want to compare two treatments—we divide experimental objects into n homogeneous pairs (to exclude effects due to the inhomogeneity of the data) and apply the new treatment to one unit of the pair while the other one is control. Denote $Z_1, \ldots, Z_n$ control observations and $Y_1, \ldots, Y_n$ treatment observations and define $X_i = Y_i - Z_i$ for $i = 1, \ldots, n$ and assume that the assumptions above hold. Testing hypothesis $\mathbf{H}_0$ in this case is equivalent testing if there is no effect of the new treatment, while the alternative $\Delta > 0$ means the positive effect of the new treatment.

3 Rank tests of symmetry with additive measurement errors

Now suppose that we observe instead of X_i random variables $W_i = X_i + V_i$, $i = 1, \ldots, n$, where V_i are i.i.d. random variables independent on $X_1, \ldots, X_n$ with an unknown continuous density $g(v)$ symmetric around 0 with finite Fisher information. We still want to test the hypothesis $\mathbf{H}_0 \colon \Delta = 0$ which corresponds to the distribution of X_i, but these are not observable. Analogously as in the previous section define

$$\tilde{S}_n = \sum_{i=1}^{n} a_n^+(\tilde{R}_i^+) \operatorname{sign}(W_i),$$

where $\tilde{R}_i^+$ are the ranks of $|W_i|$ among $|W_1|, \ldots, |W_n|$.

Denote $h(w) = \int_{-\infty}^{\infty} f(w-v)g(v)dv$ density of W_i. Immediately this density is symmetric around Δ, particularly under null hypothesis $h(w)$ is symmetric around zero, hence the exact distribution of $\tilde{S}_n$ is the same as the distribution of S_n in model without measurement errors, as well as the asymptotic distribution is also normal with mean 0 and variance σ_n^2. It means that if we use the critical region for the test without measurement errors for the case with measurement errors, we get the same size α.

However, the distribution of $\tilde{S}_n$ under $\mathbf{K}_n$ is slightly different, it is asymptotically normal with mean $\tilde{\mu}_n = \sqrt{n}\Delta^*\gamma(\varphi^+, h)$ and variance σ_n^2, asymptotic relative efficiency (ARE) of this test in measurement errors model relative to the same test in model without measurement errors is (ratio of respective noncentrality parameters):

$$\mathrm{ARE}(\tilde{S}_n, S_n) = \left(\frac{\tilde{\mu}_n}{\mu_n}\right)^2 = \frac{\gamma^2(\varphi^+, h)}{\gamma^2(\varphi^+, f)} = \left(\frac{\int_{\frac{1}{2}}^{1} \varphi(u)\varphi(u,h)du}{\int_{\frac{1}{2}}^{1} \varphi(u)\varphi(u,f)du}\right)^2 .$$

Note that the number $\mathrm{ARE}(\tilde{S}_n, S_n) \cdot n$ can be interpreted (Pitman's interpretation of ARE) as a number of observations which we would need for reaching asymptotic the same power as by using the same test in the case without measurement errors.

The following example is an exercise in the book [7, p. 126], but the original data are from the article [11].

Example 1 In a study of the comparative tensile strength of tape-closed and sutured wounds, the following results were obtained on 10 rats, 40 days after incisions made on their backs had been closed by suture or by surgical tape. Results are summarised in the table 4.

Rat	1	2	3	4	5	6	7	8	9	10
Tape	659	984	397	574	447	479	676	761	647	577
Suture	452	587	460	787	351	277	234	516	577	513
Difference	207	397	−63	−213	96	202	442	245	70	64

Table 4: Comparative tensile strength (lb. per sq. in.) of tape-closed and sutured wounds of rats.

We test the hypothesis of no difference between tape-closed and sutured wounds against the alternative that the tape-closed wounds are stronger. We illustrate it by the Wilcoxon scores $\varphi(u) = u - 1/2$. As the sample size is small ($n = 10$), we use the exact distribution of S_n. The p-value of this test is 0.0244—at the level of significance $\alpha = 0.05$ we reject the hypothesis of no difference between this two treatments.

Now suppose that we observe the differences of both treatments with a measurement error. We make 10 000 replications, every time we contaminate the original data with random errors and compute respective p-value. Finally we estimate the p-value as the mean of p-values of these 10 000 replications (the p-value is the probability of obtaining a test statistic at least as extreme as the one that was actually observed). Results are summarised in the table 5 ($N(0, b)$ stands for the normal distribution with variance b, U is uniform and $C(0, b)$ is Cauchy with scale parameter b).

Error	0	N(0,100)	N(0,400)	U(−30,30)	C(0,5)	C(0,20)
p-value	0.0244	0.0271	0.0285	0.0283	0.0346	0.0564

Table 5: Effect of the measurement errors on p-value of Wilcoxon signed-rank test.

We can see that the measurement errors increase the p-value and even errors with large variance may mask effect of the new treatment, so that we would not reject the hypothesis $\mathbf{H}_0$.

Now suppose that the measurement errors V_i are symmetric around $\Delta_0 \neq 0$. This case corresponds to the situation where the measurement is affected by a systematic error (for example caused by wrong calibration, or by human factor).

If we use the classical test based on $\tilde{S}_n$ and the critical value for the test without measurement errors (we ignore the measurement errors) in this situation, the probability of error of the first kind will be no longer α (it can increase or decrease). The asymptotic probability of error of the first kind of the test is

$$\alpha^* = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \frac{\mu_n(\Delta_0)}{\sigma_n}\right),$$

with $\mu_n(\Delta_0) = n\Delta_0\gamma(\varphi^+, h)$ and $\Phi(x)$ standard normal distribution function. From the previous formula we can see that if $\Delta_0 < 0$, then $\alpha^* < \alpha$ (the real value is lower than prescribed) and if $\Delta_0 > 0$, then $\alpha^* > \alpha$ (the real value is greater than prescribed). For the opposite alternative $\Delta < 0$ is the situation symmetric and for the both-sided alternative the real value is always greater than prescribed.

4 R-estimates of location parameter

Let us start with the model without measurement errors: Let $X_1, \ldots, X_n$ be i.i.d. random variables with an unknown continuous distribution function $F(x - \Delta)$ and continuous density $f(x - \Delta)$, where f is symmetric around zero. Moreover, suppose that X_i have finite variance and finite Fisher information $I(f)$. We want to estimate the location parameter Δ. Again, as in Section 2 the following theoretical results are summarizing existing results about R-estimates in location model; detailed proofs may be found in books [3], [5].

Now, define

$$S_n(t) = \sum_{i=1}^{n} a_n^+(R_i^+(t))\,\mathrm{sign}(X_i - t),$$

where $R_i^+(t)$ are the ranks of $|X_i - t|$ among $|X_1 - t|, \ldots, |X_n - t|$ and the approximate scores $a_n^+(i)$ are defined the same way as in Section 2. As an estimator of Δ it is proposed the value of t which solves the equation $S_n(t) = 0$. As $S_n(t)$ is discontinuous, such an equation may have no solution; then we define the R-estimator as

$$\hat{\Delta}^{(R)}_{n,(X)} = \hat{\Delta}^{(R)}_n = \frac{1}{2}\left[\sup\{t\colon S_n(t) > 0\} + \inf\{t\colon S_n(t) < 0\}\right].$$

In general, the value of $\hat{\Delta}^{(R)}_n$ cannot be expressed by some formula, that's why we have to use some numerical method for finding $\hat{\Delta}^{(R)}_n$, but for special choices of the scores a_n^+ we can get the accurate expression. If $a_n^+(i) = 1$ for all $i = 1, \ldots, n$, then

$$\hat{\Delta}^{(R)}_n = \hat{\Delta}^{(\text{med})}_n = \text{med}\{X_1, \cdots, X_n\}$$

and if $a_n^+(i) = \frac{i}{n+1}$ we have

$$\hat{\Delta}^{(R)}_n = \hat{\Delta}^{(H-L)}_n = \text{med}\left\{\frac{X_i + X_j}{2}, 1 \le i \le j \le n\right\}$$

(Hodges–Lehmann estimator).

Under assumptions mentioned above from asymptotic representation for $\hat{\Delta}^{(R)}_n$ (see [5, p. 244]) we get the asymptotic distribution of $\hat{\Delta}^{(R)}_n$:

$$\sqrt{n}(\hat{\Delta}^{(R)}_n - \Delta) \xrightarrow{d} N\left(0, \frac{A^2(\varphi^+)}{\gamma^2(\varphi^+, f)}\right).$$

Hence we can also express the asymptotic confidence interval for Δ, this confidence interval has one disadvantage – we do not know the density f. For practical computations it is necessary to estimate it. Anyway, it also exists another approach based on the distribution of $S_n(t)$. Let $C_{n,\alpha}$ be the smallest value for which following inequality holds

$$P_{\mathbf{H}_0}(|S_n(0)| \le C_{n,\alpha}) \ge 1 - \alpha.$$

We can compute $C_{n,\alpha}$ from the exact distribution of S_n, or we can use the asymptotic normal approximation mentioned in Section 2. Hence we get

$$C_{n,\alpha} \sim \sqrt{n}A(\varphi^+)\Phi^{-1}(1 - \alpha/2).$$

And finally define the confidence interval $\left(\hat{\Delta}^{(R)}_{L,n}, \hat{\Delta}^{(R)}_{U,n}\right)$:

$$\hat{\Delta}^{(R)}_{L,n} = \frac{1}{2}\left[\sup\{t\colon S_n(t) > C_{n,\alpha}\} + \inf\{t\colon S_n(t) < C_{n,\alpha}\}\right],$$

$$\hat{\Delta}^{(R)}_{U,n} = \frac{1}{2}\left[\sup\{t\colon S_n(t) > -C_{n,\alpha}\} + \inf\{t\colon S_n(t) < -C_{n,\alpha}\}\right].$$

5 R-estimates of location parameter under additive measurement errors

Now suppose that we observe instead of X_i random variables $W_i = X_i + V_i$, $i = 1, \ldots, n$, where V_i are i.i.d. random variables independent with $X_1, \ldots, X_n$ with an unknown continuous density $g(v)$ symmetric around 0 with finite Fisher information and finite variance. We still want to estimate the parameter Δ. Recall that $h(w) = \int_{-\infty}^{\infty} f(w-v)g(v)dv$ is density of W_i.

Now, define

$$\tilde{S}_n(t) = \sum_{i=1}^{n} a_n^+(\tilde{R}_i^+(t)) \operatorname{sign}(W_i - t),$$

where $\tilde{R}_i^+(t)$ are the ranks of $|W_i - t|$ among $|W_1 - t|, \ldots, |W_n - t|$ and an estimator of Δ is

$$\hat{\Delta}_{n,(W)}^{(R)} = \frac{1}{2}\left[\sup\{t\colon \tilde{S}_n(t) > 0\} + \inf\{t\colon \tilde{S}_n(t) < 0\}\right]. \quad (1)$$

From asymptotic representation for $\hat{\Delta}_{n,(W)}^{(R)}$ (see [5, p. 244]) we get the asymptotic distribution of $\hat{\Delta}_{n,(W)}^{(R)}$:

$$\sqrt{n}(\hat{\Delta}_{n,(W)}^{(R)} - \Delta) \xrightarrow{d} N\left(0, \frac{A^2(\varphi^+)}{\gamma^2(\varphi^+, h)}\right).$$

It means that if we replace in formula for model without measurement errors f with h or use the same setup for confidence interval for model without measurement errors, we get the result for R-estimate in measurement errors model. The ARE $\hat{\Delta}_{n,(W)}^{(R)}$ relative to $\hat{\Delta}_{n,(X)}^{(R)}$ (R-estimate in model with measurement errors relative to R-estimate in model without measurement errors) is

$$\operatorname{ARE}\left(\hat{\Delta}_{n,(W)}^{(R)}, \hat{\Delta}_{n,(X)}^{(R)}\right) = \frac{\gamma^2(\varphi^+, h)}{\gamma^2(\varphi^+, f)} = \left(\frac{\int_{\frac{1}{2}}^{1} \varphi(u)\varphi(u,h)du}{\int_{\frac{1}{2}}^{1} \varphi(u)\varphi(u,f)du}\right)^2.$$

Now suppose that the measurement errors V_i are symmetric around $\Delta_0 \neq 0$. Because of the form of W_i we cannot distinguish what part of W_i "belongs" to Δ and what to Δ_0. We try to consider the R-estimate given by the equation (1). We have to realize that the estimate $\hat{\Delta}_{n,(W)}^{(R)}$ does not estimate parameter Δ, but $\Delta + \Delta_0$, hence $\hat{\Delta}_{n,(W)}^{(R)}$ is not consistent estimate of Δ. Applying asymptotic representation theorem for $\hat{\Delta}_{n,(W)}^{(R)}$ we have

$$\sqrt{n}(\hat{\Delta}_{n,(W)}^{(R)} - \Delta) \xrightarrow{d} N\left(\Delta_0, \frac{A^2(\varphi^+)}{\gamma^2(\varphi^+, h)}\right).$$

We have also made a simulation study to show how the estimates perform for finite sample situation; because of its importance, it will be an object of a separate study. Anyway, we should mention at least some notes. We have compared mean, median, Hodges–Lehmann estimator and the R-estimate based on the score function $\varphi(u) = \Phi^{-1}(u)$. The simulation study indicates that all considered estimators estimate parameter Δ approximately the same except from the mean in heavy-tailed distributions. If the original distribution of X_i or the distribution of errors V_i is heavy-tailed, then mean fails. Much more interesting is the comparison of variances of R-estimates. It depends on the choice of score function φ and the distributions of measurement errors. It increases with increasing variance of measurement errors. Hodges–Lehmann estimator and R-estimator based on normal scores attain the best accuracy for short-tailed distributions, median attains the best accuracy for heavy-tailed distributions, but in general we can say that Hodges–Lehmann estimator attains the best accuracy among other estimates for any measurement errors. Anyway, if measurement errors do not have large variance, the classical R-estimates (rank tests) may be used quite effectively.

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References

[1] Jurečková, J., Kalina, J., Picek, J., Saleh, A. K. Md. E.: *Rank tests of linear hypothesis with measurement errors both in regressors and responses.* In: Preprint of Department of Probability and Mathematical Statistics *Charles University*, Prague, **66** (2009).

[2] Jurečková, J., Picek, J.: *Rank tests in partially linear and measurement errors models.* Submitted.

[3] Jurečková, J., Picek, J.: Robust Statistical Methods with R. *Chapman and Hall/CRC*, Boca Raton, 2005.

[4] Jurečková, J., Picek, J., Saleh, A. K. Md. E.: *Rank tests and regression rank score tests in measurement error models.* Computational Statistics and Data Analysis **54** (2010), 3108–3120.

[5] Jurečková, J., Sen, P. K.: Robust Statistical Procedures: Asymptotics and Interrelations. *Wiley*, New York, 1996.

[6] Hájek, J., Šidák, Z., Sen, P. K.: Theory of rank tests. Second Edition. *Academic Press*, New York, 1999.

[7] Lehmann, E. L.: Nonparametrics: Statistical Methods Based on Ranks. *Holden Day*, San Francisco, 1975.

[8] Navrátil, R.: *Rank Tests of Symmetry in Measurement Errors Models.* In: WDS'10 Proceedings of Contributed Papers: Part I – Mathematics and Computer Sciences (2010), 177–182.

[9] Saleh, A. K. Md. E., Picek, J., Kalina, J.: *Nonparametric Estimation of Regression Parameters in Measurement Error Models.* Metron **68** (2010), 177–200.

[10] Saleh, A. K. Md. E., Picek, J., Kalina, J.: *R-estimation of the parameters of a multiple regression model with measurement errors.* Metrika doi:10.1007/s00184-010-0328-2 (2010).

[11] Ury, H. K., Forrester, J. C.: *More on Scale Dependence of Wilcoxon Signed-rank Test.* The American Statistician **24** (1970), 25–26.