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The Projectivization of Conformal Models of Fibrations Determined by the Algebra of Quaternions*

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Abstract

Our aim is to study the principal bundles determined by the algebra of quaternions in the projective model. The projectivization of the conformal model of the Hopf fibration is considered as example.

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1 Introduction

A. P. Norden developed the theory of normalization which appeared useful in applications to conformal, non-Euclidean and linear geometry, [7]. By means of the normalization theory, A. P. Shirokov [11] succeeded to construct conformal models of non-Euclidean spaces. We show here basic steps of this construction.

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Let a real non-degenerate hyperquadric Q be given in the projective space $\mathbb{P}^{n+1}$. Let us choose a projective frame $(E_0, \dots, E_{n+1})$ such that E_{n+1} is the pole of the hyperplane $y^{n+1} = 0$, and the straight line $E_n E_{n+1}$ intersects the hyperquadric Q in two real points N and N', and the points $E_0, \dots, E_{n-1}$ belong to the polar of the straight line $E_n E_{n+1}$.

Then the analytic expression of the hyperquadric Q reduces to the form

$$\mathbf{y}^2 = a_{pq} y^p y^q + (y^n)^2 - (y^{n+1})^2 = 0, \tag{1}$$

where $p, q = 0, \dots, n-1$. The hyperquadric (1) divides the space $\mathbb{P}^{n+1}$ into the inner part characterized by $\mathbf{y}^2 < 0$, and the outer part $\mathbf{y}^2 > 0$, and intersects the hyperplane $y^{n+1} = 0$ in a hypersphere $\tilde{Q}$

$$a_{pq} y^p y^q + (y^n)^2 = 0$$

which can be either real or imaginary.

Let us construct the stereographic projection with the pole $N(0\colon \dots \colon 0\colon 1\colon 1)$ of the hyperplane $\mathbb{P}^n \colon y^{n+1} = 0$ into the hyperquadric Q. If $U(y^0\colon \dots \colon y^n\colon 0) \in \mathbb{P}^n$ take the straight line

$$\lambda U + \mu N = (\lambda y^0\colon \dots \colon \lambda y^{n-1}\colon \lambda y^n + \mu\colon \mu);$$

coordinates of its intersection point with Q satisfy

$$\lambda^2 a_{pq} y^p y^q + (\lambda y^n + \mu)^2 - \mu^2 = 0, \quad \lambda \neq 0.$$

Setting $k = \frac{\mu}{\lambda}$ we can write the previous equation as

$$a_{pq} y^p y^q + (y^n)^2 + 2k y^n = 0.$$

Let us distinguish two possibilities.

1) If $y^n \neq 0$, i. e. the point $U \notin \mathbb{P}^{n-1}$, then

$$k = -\frac{a_{pq} y^p y^q + (y^n)^2}{2y^n}.$$

Hence the intersection point of the straight line UN with the hyperquadric Q is uniquely determined.

2) If $y^n = 0$ then $a_{pq} y^p y^q = 0$ holds. The intersection $\tilde{Q}$ with the $(n-1)$-plane $y^n = 0$ is an ideal hyperplane $\mathbb{P}^{n-1}$ of the hyperplane $y^{n+1} = 0$. In this case, the intersection point of the straight line UN with the hyperquadric Q is not uniquely determined. The straight line UN is in the tangent plane $T_N \colon y^n - y^{n+1} = 0$ of the point N.

Hence examine only the first case $y^n \neq 0$. In the hyperplane $y^{n+1} = 0$, consider the $(n-1)$-plane $P^{n-1} \colon y^n = 0$ as an ideal hyperplane; we obtain the structure of affine space $\mathbb{A}^n$ on the rest. In $\mathbb{A}^n$, we can introduce Cartesian coordinates $u^i = y^i / y^n$. Moreover, in $\mathbb{A}^n$ there exists the structure of Euclidean space $\mathbb{E}^n$ with the metric form

$$ds_0^2 = \pm a_{pq} du^p du^q. \tag{2}$$

In this case, the point $U(u^0\colon u^1\colon \dots \colon u^{n-1}\colon 1\colon 0)$ is mapped into the point

$$X_1(2u^0\colon \dots \colon 2u^{n-1}\colon 1 - a_{pq}u^pu^q\colon -1 - a_{pq}u^pu^q).$$

Let us normalize the hyperquadric (1) self-polar, taking the lines of the sheaf of lines with a fixed center $Z = E_{n+1}$ as normals of the first-order, and their polar $(n-1)$-planes belonging to the hyperplane $y^{n+1} = 0$ as second-order normals. The straight line $E_{n+1}X_1$ intersects the hyperplane $y^{n+1} = 0$ in the point

$$X(2u^0\colon \dots \colon 2u^{n-1}\colon 1 - a_{pq}u^pu^q\colon 0).$$

Note that the polar of the point X related to the hyperquadric (1) intersects the hyperplane $y^{n+1} = 0$ exactly in the $(n-1)$-dimensional second-order normal which corresponds to the first-order normal X_1E_{n+1}. Hence in the hyperplane $y^{n+1} = 0$, a point X in general position is in correspondence with an $(n-1)$-plane, and the hyperplane $y^{n+1} = 0$ appears to be a polary normalized projective space $\mathbb{P}^n$ with the same geometry as the quadric itself.

Let us define a second-order normal by basic points $Y_i = \partial_i X - l_i X$. We find the scalar product $(X, X) = (1 + a_{pq}u^pu^q)^2$. The points X and Y_i are polar conjugate, i.e. the scalar product $(X, Y_i) = 0$. From these conditions we calculate the normalizator l_i:

$$l_i = \frac{2a_{is}u^s}{1 + a_{pq}u^pu^q}.$$

The decompositions

$$\partial_j Y_i = l_j Y_i + \Gamma^s_{ij} Y_s + p_{ij} X$$

determine components of the projective-Euclidean connection Γ^k_{ij} and the tensor p_{ij} [7]. Then the differential equations of the normalized space $\mathbb{P}^n\colon y^{n+1} = 0$ take the form

$$\partial_i X = Y_i + l_i X, \qquad \nabla_j Y_i = l_j Y_i + p_{ij} X. \tag{3}$$

Covariant differentiation of the equation $(X, Y_i) = 0$ gives

$$(\partial_j X, Y_i) + (X, \nabla_j Y_i) = 0.$$

Hence, by (3) we get

$$(X, \nabla_j Y_i) = -(\partial_j X, Y_i) = -(Y_j, Y_i) - l_j(X, Y_i)$$
$$-(\partial_i X - l_i X, \partial_j X - l_j X) = -(\partial_i X, \partial_j X) - l_i l_j (X, X).$$

Therefore

$$p_{ij} = \frac{(X, \nabla_j Y_i)}{(X, X)} = -\frac{(\partial_i X, \partial_j X)}{(X, X)} + l_i l_j = -\frac{4a_{ij}}{(1 + a_{pq}u^pu^q)^2}. \tag{4}$$

Hence considering in $\mathbb{A}^n$ the structure of the Euclidean space $\mathbb{E}^n$ with the Cartesian coordinates u^i we obtain a conformal model of a polar normalized projective space $\mathbb{P}^n$, i.e. a non-Euclidean space with the metric tensor

$$ds^2 = g_{ij}du^i du^j = \frac{\pm a_{ij}du^i du^j}{(1 + a_{pq}u^pu^q)^2}. \tag{5}$$

As we can see from (2) and (5), the obtained non-Euclidean space is conformally equivalent to the Euclidean space.

Quadrics in the projective spaces of a special type have been also studied in [2], [3].

2 The projectivization of conformal models of fibrations determined by the algebra of quaternions

Assume the associative unital 4-dimensional algebra $\mathbb{A}$ of quaternions [8], [9] with the basis $1, i, j, k$ and the multiplication table

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

As well known, any quaternion can be uniquely expressed as $\mathbf{x} = x^0 + x^1 i + x^2 j + x^3 k$, conjugation is given by $\mathbf{x} \mapsto \bar{\mathbf{x}} = x^0 - x^1 i - x^2 j - x^3 k$, $\overline{\mathbf{xy}} = \bar{\mathbf{y}}\bar{\mathbf{x}}$ holds, the number $\mathbf{x}\bar{\mathbf{x}} = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ is real, and $\mathbf{x} \mapsto |\mathbf{x}| = \sqrt{\mathbf{x}\bar{\mathbf{x}}}$ defines a norm corresponding to the scalar product $\mathbf{xy} = \frac{1}{2}(\mathbf{x}\bar{\mathbf{y}} + \mathbf{y}\bar{\mathbf{x}})$ that turns $\mathbb{A}$ into the four-dimensional Euclidean space $\mathbb{E}^4$. Since $|1| = |i| = |j| = |k| = 1$ the basis elements are called units. For any $\mathbf{x}$ with $|\mathbf{x}| \neq 0$ there exists the inverse element $\mathbf{x}^{-1} = \frac{\bar{\mathbf{x}}}{|\mathbf{x}|^2}$. The set of all invertible elements from $\mathbb{A}$

$$\tilde{\mathbb{A}} = \{\mathbf{x} \mid \mathbf{x} \neq 0\}$$

is a Lie group [10].

The group of quaternions of the unit norm $\mathbf{x}\bar{\mathbf{x}} = 1$ can be interpreted as the unit sphere S^3

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1 \tag{6}$$

in the Euclidean space $\mathbb{E}^4$.

We extend $\mathbb{E}^4$ into $\mathbb{P}^4$; taking

$$u^0 = \frac{y^0}{y^4}, \quad u^1 = \frac{y^1}{y^4}, \quad u^2 = \frac{y^2}{y^4}, \quad u^3 = \frac{y^3}{y^4}$$

we introduce homogeneous coordinates $(y^0 \colon y^1 \colon y^2 \colon y^3 \colon y^4)$. The quadric S^3 has coordinate expression

$$\mathbf{y}^2 = (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 - (y^4)^2 = 0. \tag{7}$$

The quadric (7) divides $\mathbb{P}^4$ into the inner part characterized by $\mathbf{y}^2 < 0$ and the outer part $\mathbf{y}^2 > 0$, and intersects the hyperplane $y^0 = 0$ in the two-sphere S^2

$$(y^1)^2 + (y^2)^2 + (y^3)^2 - (y^4)^2 = 0.$$

The point E_0 of the projective frame $(E_0, \dots, E_4)$ is the pole of the hyperplane $y^0 = 0$, the straight line E_0E_4 intersects the quadric in two real points $N(1\colon 0\colon 0\colon 0\colon 1)$ and $N'(-1\colon 0\colon 0\colon 0\colon 1)$, and the points E_1, E_2, E_3 belong to the polar $\mathbb{P}^2$ of the straight line E_0E_4.

The tangent plane at the point N has the equation $y^0 - y^4 = 0$. It intersects $\mathbb{P}^3$ in the mentioned 2-plane $\mathbb{P}^2$: $y^4 = 0$. Hence in the hyperplane $y^0 = 0$ there is a structure of affine space $\mathbb{A}^3$ for which $\mathbb{P}^2$ is the plane at infinity. Consequently, under the assumption $y^4 \neq 0$ we can introduce Cartesian coordinates

$$u^i = \frac{y^i}{y^4}, \quad i = 1, 2, 3.$$

Moreover, the sphere S^2 determines in $\mathbb{A}^3$ the structure of Euclidean space $\mathbb{E}^3$ with the metric form

$$ds_0^2 = (du^1)^2 + (du^2)^2 + (du^3)^2. \tag{8}$$

Assume the stereographic projection of the hyperplane $y^0 = 0$ from the pole $N(1\colon 0\colon 0\colon 0\colon 1)$ onto the quadric (7). The point $U(0\colon u^1\colon u^2\colon u^3\colon 1)$ is mapped into the point

$$X_1(-1 + r^2\colon 2u^1\colon 2u^2\colon 2u^3\colon 1 + r^2),$$

that is, the Cartesian coordinates u^i are local coordinates on the quadric (7). Here $r^2 = (u^1)^2 + (u^2)^2 + (u^3)^2$ is the quadrat of distance of the point U from the origin of the Euclidean metric (8).

Let us normalize the quadric (7) self-polar, taking as the first-order normals straight lines passing through E_0, and as second order polars their polar two-planes belonging to the hyperplane $y^0 = 0$ as second order normals. The straight line E_0X_1 intersects the hyperplane $y^0 = 0$ in the point

$$X(0\colon 2u^1\colon 2u^2\colon 2u^3\colon 1 + r^2).$$

It is more suitable to take the point $X \in \mathbb{P}^3$ instead of the point X_1. Note that the polar of the point X related to the quadric (7) intersects the hyperplane $y^0 = 0$ in that two-dimensional normal of the second order which corresponds to the normal of the first order X_1E_0. Hence in the hyperplane $y^0 = 0$, a point X in general position corresponds to a two-plane, and the hyperplane $y^0 = 0$ is the normalized projective space $\mathbb{P}^3$. Particularly, if $r^2 = 1$ the point X belongs to the sphere S^2, and its polar is a tangent 2-plane in this point.

Let us define the second order polar as a span of the points $Y_i = \partial_i X - l_i X$. Then X and Y_i are polar conjugate, i.e. $(X, Y_i) = 0$. From this condition we find coordinates of the normalizer of the space $\mathbb{P}^3$

$$l_i = \frac{(\partial_i X, X)}{(X, X)}.$$

Since $(X, X) = -(r^2 - 1)^2$ we obtain

$$l_1 = \frac{2u^1}{r^2 - 1}, \qquad l_2 = \frac{2u^2}{r^2 - 1}, \qquad l_3 = \frac{2u^3}{r^2 - 1},$$

and by (4) we have finally

$$p_{11} = p_{22} = p_{33} = \frac{4}{(r^2-1)^2}.$$

Now introducing in $\mathbb{A}^3$ the structure of Euclidean space $\mathbb{E}^3$ with u^i as Cartesian coordinates we find the conformal model C^3 with the metric form

$$ds^2 = g_{ij}du^i du^j = \frac{(du^1)^2 + (du^2)^2 + (du^3)^2}{(r^2-1)^2}. \tag{9}$$

The corresponding Riemannian (Levi–Civita) connection appears to be of constant curvature $K = 1$, non-vanishing components (Christoffel symbols) of connection are just

$$\Gamma^1_{11} = -\Gamma^1_{22} = -\Gamma^1_{33} = \Gamma^2_{21} = \Gamma^3_{13} = -\frac{2u^1}{r^2-1},$$

$$\Gamma^2_{22} = \Gamma^1_{12} = -\Gamma^2_{11} = -\Gamma^2_{33} = \Gamma^3_{32} = -\frac{2u^2}{r^2-1},$$

$$\Gamma^3_{33} = -\Gamma^3_{22} = \Gamma^1_{13} = -\Gamma^3_{11} = \Gamma^2_{23} = -\frac{2u^3}{r^2-1}.$$

As an example, we examine the fibration defined by the subalgebra of complex numbers.

3 Example

Let us write a quaternion in the form

$$\mathbf{x} = x^0 + x^1 i + (x^2 + x^3 i)j = z_1 + z_2 j, \quad z_1, z_2 \in \mathbb{R}(i),$$

where $\mathbb{R}(i)$ is a 2-dimensional subalgebra of complex numbers with basis $\{1, i\}$. The set of its invertible elements

$$\tilde{\mathbb{R}}(i) = \{\lambda = a + bi \mid \lambda \neq 0\}, \quad a, b \in \mathbb{R}$$

turns out to be a Lie subgroup of the group $\tilde{\mathbb{A}}$, a 2-plane with exception of one point.

The canonical projection $\pi\colon \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}/\tilde{\mathbb{R}}(i)$ reads

$$\pi(\mathbf{x}) = (z_1 \colon z_2).$$

The factorspace $\tilde{\mathbb{A}}/\tilde{\mathbb{R}}(i)$ is a complex projective line $P(i)$ covered by two charts

$$U_1 = \{[z_1 \colon z_2] \mid z_2 \neq 0\} \quad \text{with the coordinate } z = \frac{z_1}{z_2}$$

and

$$U_2 = \{[z_1 \colon z_2] \mid z_1 \neq 0\} \quad \text{with the coordinate } \tilde{z} = \frac{z_2}{z_1}.$$

Let the point $z = u + iv \in P(i)$ is in U_1. Then the coordinate expression of the projection π in real coordinates is

$$\pi(z_1, z_2) = z = \left(\frac{x^0x^2 + x^1x^3}{(x^2)^2 + (x^3)^2}, \frac{x^1x^2 - x^0x^3}{(x^2)^2 + (x^3)^2} \right). \tag{10}$$

If $z = \frac{z_1}{z_2}$, where in homogeneous coordinates

$$z_1 = \frac{y^0 + y^1 i}{y^4}, \quad z_2 = \frac{y^2 + y^3 i}{y^4},$$

we obtain 2-planes $L_2\colon z_1 - zz_2 = 0$ which are given by a pair of equations over reals

$$\left.\begin{aligned} y^0 - uy^2 + vy^3 &= 0, \\ y^1 - vy^2 - uy^3 &= 0. \end{aligned}\right\} \tag{11}$$

These 2-planes determine the equations of fibres in the projective space $\mathbb{P}^4$. The projection $\pi(\mathbf{y}) = z$ can be written as

$$\pi(\mathbf{y}) = \left(\frac{y^0y^2 + y^1y^3}{(y^2)^2 + (y^3)^2}, \frac{y^1y^2 - y^0y^3}{(y^2)^2 + (y^3)^2} \right),$$

which is equivalent to (10). The system (11) together with (7),

$$\begin{gathered} (y^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 - (y^4)^2 = 0, \\ y^0 - uy^2 + vy^3 = 0, \\ y^1 - vy^2 - uy^3 = 0, \end{gathered}$$

defines on this quadric a 2-parameter family of second order curves which define the fibration. Excluding y^0 from the formulas we find the projection of the family of fibres onto the base $y^0 = 0$. Passing to the Cartesian coordinates we obtain

$$\left.\begin{gathered} (u^1)^2 + (u^2)^2 + (u^3)^2 + (uu^2 - vu^3)^2 = 1, \\ u^1 - vu^2 - uu^3 = 0. \end{gathered}\right\} \tag{12}$$

There is a correspondence of these equations with the equations (21) ([4], p. 89). If $\mathbf{y}$ is a point on the quadric distinct from N (i.e. $y^0 - y^4 \neq 0$ holds), the corresponding point ξ in $\mathbb{E}^3$: $y^0 = 0$ is uniquely determined by the homogeneous coordinates $(0\colon y^1\colon y^2\colon y^3\colon y^4 - y^0)$, that is

$$\xi\left(0\colon \frac{y^1}{y^4 - y^0}\colon \frac{y^2}{y^4 - y^0}\colon \frac{y^3}{y^4 - y^0}\colon 1\right),$$

and the corresponding Cartesian coordinates in the space $\mathbb{A}^3\colon y^4 \neq 0$ are

$$x = \frac{u^1}{1 - u^0}, \quad y = \frac{u^2}{1 - u^0}, \quad z = \frac{u^3}{1 - u^0}.$$

The inverse mapping is characterized by the formulas

$$u^0 = \frac{\xi^2 - 1}{\xi^2 + 1}, \quad u^1 = \frac{2x}{\xi^2 + 1}, \quad u^2 = \frac{2y}{\xi^2 + 1}, \quad u^3 = \frac{2z}{\xi^2 + 1}, \quad \xi^2 = x^2 + y^2 + z^2$$

similar to the formulas (18) (cf. [4], p. 88). Hence the coordinates of the points $\mathbf{y}$ and ξ are related by the conformal mapping. Substituting these expressions into (12) we obtain the equations of the family of fibres in the form

$$x^2 + y^2 + z^2 - 2(uy - vz)^2 = 1,$$
$$x - vy - uz = 0.$$

These equations coincide with (21) (cf. [4], p. 89).

In the conclusion, note that the stereographic projection with the pole N' yields the Euclidean space with the same ideal 2-plane $\mathbb{P}^2$, and considering both stereographic projections together we get the covering of the sphere, the transition functions of which are given by the inverse mapping.

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