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Holomorphically Projective Mappings of Compact Semisymmetric Manifolds

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Abstract

In this paper we consider holomorphically projective mappings from the compact semisymmetric spaces A_n onto (pseudo-) Kählerian spaces $\bar{K}_n$. We proved that in this case space A_n is holomorphically projective flat and $\bar{K}_n$ is space with constant holomorphic curvature. These results are the generalization of results by T. Sakaguchi, J. Mikeš, V. V. Domashev, N. S. Sinyukov, E. N. Sinyukova, M. Škodová, which were done for holomorphically projective mappings of symmetric, recurrent and semisymmetric Kählerian spaces.

Key words: Holomorphically projective mapping, equiaffine space, affine-connected space, semisymmetric space, Riemannian space, Kählerian space.

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1 Introduction

Diffeomorphisms and automorphisms of geometrically generalized spaces constitute one of the current main directions in differential geometry. A large number of papers are devoted to geodesic, quasigeodesic, almost geodesic, holomorphically projective and other mappings (see [1], [6], [8], [9], [10], [14], [16], [18], [20], [21], [22], [23]). On the other hand, one line of thought is now the most important one, namely, the investigation of special affine-connected, Riemannian, Kählerian and Hermitian spaces.

As we know, Kählerian spaces are the special case of Hermitian spaces [23]. In many papers, holomorphically projective mappings and transformations of Hermitian spaces are studied (for example see [1], [3], [4], [10], [12], [15], [16], [13], [17], [19], [20], [21], [23]). These are special cases of F_1-planar mappings.

In [8], [10], F_1-planar mappings from the space A_n with affine connection onto a Riemannian space $\bar{V}_n$ are defined and studied.

In this paper, we present some new results obtained for holomorphically projective mappings from compact equiaffine semisymmetric A_n onto Kählerian spaces $\bar{K}_n$.

2 Holomorphically projective mappings

J. Mikeš and O. Pokorná [12] considered holomorphically projective mappings from equiaffine spaces A_n onto Kählerian spaces $\bar{K}_n$.

A space A_n with the affine connection Γ is *equiaffine* if in A_n the Ricci tensor is symmetric. A (pseudo-) Riemannian space $\bar{K}_n$ is called a *Kählerian space* if it contains, along with the metric tensor $\bar{g}$, an affine structure F satisfying the following relations

$$F^2 = -Id, \qquad \bar{g}(X, FY) + \bar{g}(Y, FX) = 0, \qquad \bar{\nabla} F = 0,$$

where $X, Y \in T\bar{K}_n$, $\bar{\nabla}$ is the connection of $\bar{K}_n$.

The following criteria from the paper [12] hold for holomorphically projective mappings from an equiaffine space A_n onto a Kählerian space $\bar{K}_n$.

Consider concrete mappings $f\colon A_n \to \bar{K}_n$, both spaces being referred to the general coordinate system x with respect to this mapping. This is a coordinate system where two corresponding points $M \in A_n$ and $f(M) \in \bar{K}_n$ have equal coordinates $x = (x^1, x^2, \ldots, x^n)$; the corresponding geometric objects in $\bar{K}_n$ will be marked with a bar. For example, ∇ and $\bar{\nabla}$ are connections on A_n and $\bar{K}_n$, respectively.

The equiaffine space A_n admits a *holomorphically projective mapping* f onto the Kählerian space $\bar{K}_n$ if and only if the following conditions in the common coordinate system x hold

$$\bar{\nabla}(X, Y) = \nabla(X, Y) + \psi(X)\, Y + \psi(Y)\, X - \psi(FX)\, FY - \psi(FY)\, FX.$$

where ψ is a linear form. We suppose that ψ is a gradient form, i.e. $\psi = d\Psi$. In this space A_n there is a complex structure F covariantly constant.

If $\psi \not\equiv 0$ then a holomorphically projective mapping is called *nontrivial*; otherwise it is said to be *trivial* or *affine*.

Hereafter we shall assume that in the equiaffine space A_n *the Ricci tensor Ric with respect to the structure F will be preserved*, i.e. $Ric(X, Y) = Ric(FX, FY)$.

In this case $\bar{P} = P$ holds, where

$$P^h_{ijk} = R^h_{ijk} - \frac{1}{n+2}\, (\delta^h_k R_{ij} - \delta^h_j R_{ik} + (F^h_j R_{\alpha k} - F^h_k R_{\alpha j}) F^\alpha_i + 2\, F^h_i R_{\alpha k} F^\alpha_j)$$

is the *tensor of the holomorphically projective curvature* of A_n. This tensor is an invariant object of the holomorphically projective mappings, see [10], [20], [21], [23]. This tensor is F-traceless [5].

If an equiaffine space A_n with condition $P^h_{ijk} = 0$ (*holomorphically projective flat space*) admits a holomorphically projective mapping onto a Kählerian space $\bar{K}_n$, then $\bar{K}_n$ has the constant holomorphic curvature [12].

3 Holomorphically projective mappings from semisymmetric equiaffine spaces

It is known, *semisymmetric* spaces A_n are characterized by the condition $R \circ R = 0$ [2]. These spaces were characterized by N. S. Sinyukov [20] by the following differential conditions on the Riemannian tensor: $R^h_{ijk,[lm]} = 0$. On base of the Ricci identity, this condition is written as follows

$$R^h_{\alpha jk} R^\alpha_{ilm} + R^h_{i\alpha k} R^\alpha_{jlm} + R^h_{ij\alpha} R^\alpha_{klm} - R^\alpha_{ijk} R^h_{\alpha lm} = 0.$$

We have the following [6]:

Theorem 1 *If an equiaffine semisymmetric space A_n, where the Ricci tensor R_{ij} with respect to the structure F will be preserved, admits a holomorphically projective mapping onto a Kählerian space $\bar{K}_n$, then $\bar{K}_n$ is the space with the constant holomorphically projective curvature or $\bar{K}_n$ admits a K-concircular vector field ψ_i, which satisfies*

$$\nabla_Y \psi(X) - \psi(X)\psi(Y) + \psi(FX)\psi(FY) = \frac{\Delta}{n}\, \bar{g}(X,Y), \quad \Delta = const. \tag{3.1}$$

4 Holomorphically projective mappings from compact semisymmetric equiaffine spaces

We have the following:

Theorem 2 *If a compact equiaffine semisymmetric space A_n, where the Ricci tensor R_{ij} with respect to the structure F will be preserved, admits a holomorphically projective mapping onto a Kählerian space $\bar{K}_n$ then $\bar{K}_n$ is the space with the constant holomorphically projective curvature (and K_n is holomorphically projective) or mapping is trivial (affine).*

Proof Let A_n be a compact semisymmetric space of the type described above which admits a holomorphically projective mapping onto a Kählerian space $\bar{K}_n$.

According to the Theorem 1, we have formulas (3.1) which may be written on all coordinate neighbourhood (x) in the followig form

$$\psi_{i,j} - \psi_i \psi_j + \psi_\alpha \psi_\beta F^\alpha_i F^\beta_j = \frac{\Delta}{n}\, \bar{g}_{hi}, \tag{4.1}$$

where "," is denote covariant derivative with respect to the connection ∇ on A_n, ψ_i, F^h_i and $\bar{g}_{ij}$ are components of form ψ, structure F and metric tensor $\bar{g}$, respectivelly.

Let us suppose that there exists some solution of the equation (4.1), let be denoted by φ. Using φ we obtain the following formulation of these equations:

$$\psi_{ij} \equiv \psi_{i,j} - \psi_i \varphi_j + \psi_\alpha \varphi_\beta F_i^\alpha F_j^\beta = \frac{\Delta}{n}\, \bar{g}_{hi}, \tag{4.2}$$

Since the metric tensor $\bar{g}_{ij}$ is regular $(\det(\bar{g}) \neq 0)$, in a neighbourhood of every point there exists a positive form $A^{ij} z_i z_j$ such that

$$\bar{g}_{ij}\, A^{ij} \geq 0 \quad (\text{or} \;\; \bar{g}_{ij}\, A^{ij} \leq 0).$$

Contracting formula (4.2) we have $\psi_{ij}\, A^{ij} \geq 0$ (or $\psi_{ij}\, A^{ij} \leq 0$), which may be, in detail, written in the form:

$$(\Psi_{,ij} - \Psi_{,i}\varphi_j + \Psi_{,\alpha}\varphi_\beta F_i^\alpha F_j^\beta)\, A^{ij} \geq 0 \quad (\text{or} \quad \leq 0) \tag{4.3}$$

where the symbol $\geq$ (or ≤ 0) evidently depends on the signum of the constant Δ and $\psi = d\Psi$.

The formula (4.3) may be written in the following form:

$$\frac{\partial^2 \Psi(x)}{\partial x^i \partial x^j} A^{ij}(x) + \frac{\partial \Psi(x)}{\partial x^i} B^i(x) \geq 0 \quad (\text{or} \quad \leq 0)$$

where $B^i(x)$ are function on coordinate neighbourhood of manifold.

It follows from the Hopf's Theorem (see [11], [24]) that this second order differential equations has only trivial solution $\Psi = \text{const}$. This implifies $\psi = 0$.

Herewith the proof is complete. □

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