Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **49**, 1 (2010) 33–47

Conley Type Index and Hamiltonian Inclusions*

Zdzisław DZEDZEJ

Faculty of Technical Physics and Applied Mathematics
Gdańsk University of Technology
ul. Narutowicza 11/12, 80-233 Gdańsk, Poland
e-mail: zdzedzej@mif.pg.gda.pl

(Received December 14, 2009)

Abstract

This paper is based mainly on the joint paper with W. Kryszewski [9], where cohomological Conley type index for multivalued flows has been applied to prove the existence of nontrivial periodic solutions for asymptotically linear Hamiltonian inclusions. Some proofs and additional remarks concerning definition of the index and special cases are given.

Key words: Conley index, multivalued dynamical system, Hamiltonian inclusion.

2000 Mathematics Subject Classification: 34A60, 37B30, 37J45

1 Introduction

Conley index theory (see [6]) is a topological generalization of Morse theory of critical points. Conley's original approach was developed for dynamical systems on locally compact metric spaces. In applications they where generated by vector fields in R^n ore on manifolds. There are several generalizations to the infinite dimensional case (non locally compact) e.g. by V. Benci and K. Rybakowski (comp. [21]). The properties of the index used in applications are to some extend similar to topological degree. In particular, many vector fields can be written in the form $L + H$, where L is a given linear bounded operator in some Hilbert space, and K is completely continous mapping. For a certain class of L the approach of finite dimensional approximations of Leray Schauder type has been succesfully applied by K. Gęba and his students ([13]). On the

*Supported by the Ministry of Sciences and Higher Education, Poland, under grant N N 201 394037

other hand, M. Mrozek [20] developed cohomological Conley index theory for multivalued flows in locally compact spaces. We present briefly the definition of a topological invariant called Conley index for multivalued flows generated by differential inclusions in Hilbert spaces. This generalizes the one defined in [13], [16] and is an infinite-dimensional version of the index considered in [20]. Let us mention that homotopy version of this index in $\mathbb{R}^n$ for flows generated by differential inclusions was also used by other authors (see [19], [12]).

Let us remark that we chose the cohomological definition of the index as in [20] because of its generality. The continuation is then easily proved for usc families of flows. However for flows generated by inclusions one can work also with the homotopical notions (see [19], [12]), which are, at least formally, more subtle. In a forthcoming paper [8] we prove that our index is the same up to a cohomology functor.

As an application we describe a result on the existence of nontrivial periodic solutions of asymptotically linear Hamiltonian systems with the Hamiltonian merely locally Lipschitz. This generalizes a classical theorem of H. Amann and E. Zehnder [1] (comp. e.g. [18]). The Leray -Schauder version of Conley theory seems especially well-fitted to such examples, where L is a selfadjoint operator with both positive and negative part of its' spectrum having infine-dimensional eigenspaces (finite-dimensional for a given eigenvalue). But it can be applied also in a simpler situation, where one of these two parts is finite dimensional, e.g. $L = Identity$. Definitions simlify in that case (we do not need suspension isomorphisms), i.e. our Conley index becomes the Conley index for a finite dimensional approximation with the dimension large enough. Such an approach was used in [2], [10] in a single-valued smooth case with additional assumptions. Our results on Hamiltonian inclusions can be viewed as a straightforward generalization of theorems for classical smooth systems with Hamiltonian of C^2-class to the locally Lipshitz case, where Clarke generalized gradients [5] stand in place of the usual one. Notice that results of [4], where variational methods are not exactly comparable with ours. F. Clarke uses a dual action functional and obtains its critical points as local minima. In our approach critical points of the functional are stationary points in an invariant set of an associated single-valued flow, for which it is a Lyapunov function (comp. the proof of 5.4 and 5.5).

We end with a computational example illustrating the result. Other examples can be easily produced as non-smooth perturbations of examples given in [13], [16].

2 Multivalued flows

All spaces considered here are metric.

A nonempty compact space X is *acyclic* if $H^0(X) = Z$ and $H^n(X) = 0$ for $n > 0$, where H^* denotes the Alexander–Spanier cohomology functor. A continuous mapping $f\colon X \to Y$ is called *Vietoris* iff it is proper and $f^{-1}(y)$ is acylic for every $y \in Y$.

A multivalued mapping $\varphi\colon X \multimap Y$ is *usc* iff $\varphi(x) \subset Y$ is nonempty compact, and for every open $U \subset Y$ the preimage $\varphi^{-1}(U) = \{x \in X : \varphi(x) \subset U\}$ is open in X.

An usc mapping $\varphi\colon X \multimap Y$ is *admissible* provided there exist: a space Γ, and continuous mappings $p\colon \Gamma \to X$, $q\colon \Gamma \to Y$ such that p is Vietoris and for $x \in X$ $\varphi(x) = q(p^{-1}(x))$. The main feature of Vietoris mappings is the following.

Theorem 2.1 (Vietoris, Begle) *A Vietoris map $f\colon X \to Y$ induces an isomorphism $f^*\colon H^*(Y) \to H^*(X)$.*

The class of admissible maps is a rich one: for example any acyclic map $\varphi\colon X \multimap Y$ is admissible (φ is *acyclic* if it is upper semicontinuous and, for any $x \in X$, $\varphi(x)$ is acyclic); it is determined by the pair (p_φ, q_φ) where $p_\varphi\colon \mathrm{Gr}(\varphi) \to X$ and $q_\varphi\colon \mathrm{Gr}(\varphi) \to Y$ are the restrictions of the projections $X \times Y \to X$ and $X \times Y \to Y$, respectively.

It is easy to see that the class of admissible maps is closed under superposition. If Y is a metric (real) vector space, $\varphi_1, \varphi_2\colon X \multimap Y$ are admissible and $f, g\colon X \to \mathbb{R}$ are continuous, then the linear combination $\varphi = f \cdot \varphi_1 + g \cdot \varphi_2 : X \multimap Y$ (given by $\varphi(x) := \{f(x)y_1 + g(x)y_2 \mid y_i \in \varphi_i(x),\ i = 1, 2\}$ for $x \in X$) is also admissible. For more details concerning admissible maps – see [14].

Definition 2.2 An usc mapping $\pi\colon X \times \mathbb{R} \multimap X$ is a *multivalued flow* on X provided for any $s, t \in \mathbb{R}, x, y \in X$

(i) $\pi(x, 0) = \{x\}$

(ii) $st \geq 0 \Longrightarrow \pi(x, t + s) = \pi(\pi(x, t), s)$

(iii) $y \in \pi(x, t) \Longleftrightarrow x \in \pi(y, -t)$.

(iv) the map $\varphi(x, \cdot)\colon \mathbb{R} \multimap X$ is continuous.

The flow is *admissible* iff there exists $T > 0$ such that the restriction of π to $X \times [0, T]$ is an admissible mapping in the above sense.

Let $\Delta \subseteq \mathbb{R}$. A Δ-*trajectory* is a continuous mapping $\sigma\colon \Delta \to X$ such that $\sigma(t) \in \pi(\sigma(s), t - s)$ for all $t, s \in \Delta$. The set of all Δ-trajectories in $N \subseteq X$ originating in x (i.e. $0 \in \Delta$, $\sigma(0) = x$) is denoted by $Tr_N(\Delta, x)$.

A *connection* from x to y in N is a $[0, t]$-trajectory σ in N such that $\sigma(0) = x$, $\sigma(t) = y$.

For N compact we define $\pi_N : N \times \mathbb{R} \to N$ by

$$\pi_N(x, t) = \{y\colon \mathrm{Conn}_N(t, x, y) \neq \emptyset\}.$$

Then π_N is a partial multivalued flow on N (i.e. $\pi_N(x, t)$ may be empty).

Let $A \subseteq X$. Denote

$\mathrm{Inv}\, A := \{x \in A\colon Tr_A(\mathbb{R}, x) \neq \emptyset\}$

$\mathrm{Inv}^+ A := \{x \in A\colon Tr_A(\mathbb{R}^+, x) \neq \emptyset\}$

$\mathrm{Inv}^- A := \{x \in A\colon Tr_A(\mathbb{R}^-, x) \neq \emptyset\}$

Definition 2.3 A subset $A \subseteq X$ is *invariant* (resp. *positively (negatively) invariant*) iff $\operatorname{Inv} A = A$ (resp. $\operatorname{Inv}^+ A = A$ ($\operatorname{Inv}^- A = A$)).

For a subset $A \subset X$ the set $\operatorname{Inv} A$ is a maximal invariant subset of A.

A is *strongly invariant* (*positively, negatively*) iff $\forall x \in A\ \pi(x, \mathbb{R}) \subseteq A$ ($\pi(x, \mathbb{R}^+) \subseteq A$, $\pi(x, \mathbb{R}^-) \subseteq A$ resp.).

The following is a version of the generalized Barbashin theorem.

Proposition 2.4 [9] *Let Λ be a metric space, $N \subset X$ be closed and let $\eta\colon X \times \mathbb{R} \times \Lambda \multimap X$ be a family of multivalued flows (i.e. η is upper semicontinuous and, for each $\lambda \in \Lambda$, $\eta(\cdot, \lambda) : X \times \mathbb{R} \multimap X$ is a multivalued flow). Then the graph of the set-valued map*

$$\Lambda \ni \lambda \mapsto \operatorname{Inv}(N, \eta(\cdot, \lambda))$$

is closed, i.e. for any sequence $(x_n, \lambda_n) \in N \times \Lambda$ such that $x_n \in \operatorname{Inv}(N, \eta(\cdot, \lambda_n))$, if $(x, \lambda) = \lim_{n\to\infty}(x_n, \lambda_n)$, then $x \in \operatorname{Inv}(N, \eta(\cdot, \lambda))$.

Let us remind the notion of Conley index in the case of X locally compact.

Definition 2.5 (see [20]) A *compact* set $N \subset X$ is an *isolating neighborhood* for φ if $\operatorname{Inv}(N, \varphi) \subset \operatorname{int} N$. We say that a set K invariant with respect to φ is *isolated* if there is an isolating neighborhood N such that $K = \operatorname{Inv}(N, \varphi)$.

Observe that, in view of Proposition 2.4, isolated invariant sets are compact.

Definition 2.6 A pair (P_1, P_2) of subsets of N is an *index pair* in N iff

(i) P_1, P_2 are compact and strongly positively invariant with respect to π_N,

(ii) $\operatorname{Inv}^- N \subseteq \operatorname{int}_N P_1$, $\quad \operatorname{Inv}^+ N \subseteq N \setminus P_2$,

(iii) $\operatorname{cl}(P_1 \setminus P_2) \subseteq \operatorname{int} N$.

Theorem 2.7 [20] *Let π be a multivalued flow on locally compact X and K an isolated invariant set with an isolating neighbourhood N. Then for every neighbourhood W of K there exists an index pair in N such that $\operatorname{cl}(P_1 \setminus P_2) \subseteq W$.*

Theorem 2.8 [20] *If the flow is admissible and K is an isolated invariant, then the Aleksander–Spanier cohomology groups $H^*(P_1, P_2)$ do not depend on the isolating neighborhood and on the choice of index pair.*

The graded group in Theorem 2.8 is called the *cohomological Conley index* $CH(K)$ of the set K. This index has the following properties:

Theorem 2.9 *(i) (Ważewski) If $CH(K)$ is nontrivial, then $K \neq \emptyset$.*

(ii) (Continuation) Assume that $\eta\colon X \times \mathbb{R} \times [0, 1] \multimap X$ is a family of admissible flows and let $N \subset X$ be an isolating neighbourhood for all flows $\eta(\cdot, t)$, $t \in [0, 1]^*$. Then

$$CH(\operatorname{Inv}(N, \eta(\cdot, 0))) = CH(\operatorname{Inv}(N, \eta(\cdot, 1))).$$

*One easily sees that the parameter space can be any compact metric path connected space instead of [0,1].

(iii) (Additivity) Let K_1, K_2 be disjoint isolated invariant sets for an admissible flow φ. Then $CH(K_1 \cup K_2) = CH(K_1) \oplus CH(K_2)$.

Example If $f\colon \mathbb{R}^n \multimap \mathbb{R}^n$ is usc with compact convex images and of sublinear growth, then solutions of the differential inclusion

$$x'(t) \in f(x(t))$$

form an admissible flow in $\mathbb{R}^n$. This follows from the Aronszajn type theorem that the set of solutions to a Cauchy problem is R_δ-set, in particular it is acyclic (see e.g. [14]).

3 L-flows in Hilbert spaces

Let $\mathbb{H} = (\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and $L\colon \mathbb{H} \to \mathbb{H}$ a linear bounded operator with spectrum $\sigma(L)$. We assume the following

- $\mathbb{H} = \bigoplus_{k=0}^{\infty} \mathbb{H}_k$ with all subspaces $\mathbb{H}_k$ being mutually orthogonal and of finite dimension;
- $L(\mathbb{H}_0) \subset \mathbb{H}_0$ where $\mathbb{H}_0$ is the invariant subspace of L corresponding to the part of spectrum $\sigma_0(L) = i\mathbb{R} \cap \sigma(L)$ lying on the imaginary axis,
- $L(\mathbb{H}_k) = \mathbb{H}_k$ for all $k > 0$,
- $\sigma_0(L)$ is isolated in $\sigma(L)$, i.e. $\sigma_0(L) \cap \mathrm{cl}(\sigma(L) \setminus \sigma_0(L)) = \emptyset$.

Definition 3.1 A multivalued flow $\varphi\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is called an *L-flow* if it has the form

$$\varphi(x,t) = e^{tL}x + U(t,x),$$

where $U\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is an admissible map which is completely continuous.

Let Λ be a metric space. By a *family of L-flows* we understand a set-valued map $\eta\colon \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ of the form

$$\eta(x,t,\lambda) = e^{tL}x + U(x,t,\lambda),$$

where $U\colon \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ is an admissible completely continuous mapping, such that, for each $\lambda \in \Lambda$, $\eta(\cdot,\lambda)\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is a multivalued flow.

It is clear that if $\eta\colon \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ is a family of L-flows, then, for each $\lambda \in \Lambda$, $\eta(\cdot,\lambda)\colon \mathbb{H} \times \mathbb{R} \multimap \mathbb{H}$ is an L-flow. Moreover each L-flow is an admissible flow.

Proposition 3.2 *Let $\eta\colon \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ be a family of L-flows. If $X \subset H$ is bounded and closed then $S = \mathrm{Inv}(X \times \Lambda, \eta) = \{(x,\lambda)\colon x \in \mathrm{Inv}(X)\}$ is a compact subset of $X \times \Lambda$.*

Definition 3.3 A bounded and closed subset $X \subset \mathbb{H}$ is an *isolating neighbourhood* for a flow π if $\mathrm{Inv}(X) \subset \mathrm{int}(X)$.

Proposition 3.4 *If $X \subset \mathbb{H}$ is closed and bounded, then the set-valued map $\Lambda \ni \lambda \mapsto \mathrm{Inv}(X, \eta_\lambda) \subset X$ is usc and has compact (possibly empty) values.*

Proof In view Proposition 2.4, it is sufficient to show that, given a sequence (λ_n, x_n) in $\Lambda \times X$ such that $x_n \in \mathrm{Inv}(X, \eta_{\lambda_n})$ and $\lambda_n \to \lambda_0 \in \Lambda$, (x_n) has a convergent subsequence, i.e., the set $S := \{x_n\}_{n=1}^{\infty}$ is relatively compact. Suppose it is not so.

Denote by $\mathbb{H}_-$ (resp. $\mathbb{H}_+$) the closed L-invariant subspace corresponding to the part of the spectrum $\sigma(L)$ of L with negative (resp. positive) real part. In view of the above assumptions, $\mathbb{H}$ splits into the direct sum $\mathbb{H} = \mathbb{H}_- \oplus \mathbb{H}_0 \oplus \mathbb{H}_+$. Let $P_\pm \colon \mathbb{H} \to \mathbb{H}_\pm$ and $P_0 \colon \mathbb{H} \to \mathbb{H}_0$ be the orthogonal projections. Since $\sigma_0(L)$ is isolated in $\sigma(L)$, for each $\varrho > 0$, there is $t_0 > 0$ such that, for all $x \in \mathbb{H}_+$ and $t \geq t_0$

$$\|e^{tL}x\| \geq \varrho \|x\| \tag{3.1}$$

and, for $x \in \mathbb{H}_-$ and $t \leq -t_0$,

$$\|e^{tL}x\| \geq \varrho \|x\|. \tag{3.2}$$

Clearly $S \subset \mathrm{cl}\, P_-(S) \times \mathrm{cl}\, P_0(S) \times \mathrm{cl}\, P_+(S)$. The set $\mathrm{cl}\, P_0(S)$ is compact as a closed bounded subset of a finite-dimensional space $\mathbb{H}_0$. Therefore either $\mathrm{cl}\, P_-(S)$ or $\mathrm{cl}\, P_+(S)$ is noncompact. Assume that $P_+(S)$ is not relatively compact. Hence there exists an $\varepsilon > 0$ such that $P_+(S)$ does not admit a finite ε-net and we can choose a sequence $(x_i) \in S$ such that $z_i := P_+(x_i)$, $i \geq 1$, satisfy $\|z_i - z_j\| \geq \varepsilon$ whenever $i \neq j$. Choose $\delta > 0$ and $t_0 > 0$ such that $X \subset B(0, \delta)$ and the inequality (3.1) holds for $\varrho = \frac{3\delta}{\varepsilon}$. For $i \geq 1$, set $u_i := e^{t_0 L} x_i$ and take an arbitrary $v_i \in U(x_i, t_0, \lambda_n)$; then

$$u_i + v_i \in e^{t_0 L} x_i + U(x_i, t_0, \lambda_i) = \eta_{\lambda_i}(x_i, t_0) \subset X \subset B(0, \delta).$$

Thus, for $i \neq j$,

$$3\delta \leq \|u_i - u_j\| \leq \|u_i + v_i\| + \|v_i - v_j\| + \|u_j + v_j\| < 2\delta + \|v_i - v_j\|$$

and, consequently,

$$\|v_i - v_j\| > \delta.$$

But, for each $i \geq 1$, v_i belongs to the set $\bigcup_{j=1}^{\infty} U(x_j, t_0, \lambda_j)$ being relatively compact in view of the complete continuity of U. Thus (v_i) has a convergent subsequence: a contradiction. □

Proposition 3.5 *Let Λ be a compact metric space let $\eta \colon \mathbb{H} \times \mathbb{R} \times \Lambda \multimap \mathbb{H}$ be a family of L-flows. If X is an isolating neighbourhood for some η_{λ_0} then it is an isolating neighbourhood for λ in some open neighbourhood V of λ_0 in Λ.*

Definition 3.6 An usc mapping $f \colon \mathbb{H} \multimap \mathbb{H}$ is an *L-vector field* if it is of the form $f(x) = Lx + K(x)$, where $K \colon \mathbb{H} \multimap \mathbb{H}$ is completely continuous with compact convex values, and if f induces an L-flow π on H.

Denote by $P_n\colon \mathbb{H} \to \mathbb{H}$ the orthogonal projection onto $H^n = \bigoplus_{k=0}^{n} H_k$.
Define

$$f_n\colon H^n \multimap H^n \text{ by } f_n(x) = Lx + P_n(K(x))$$

and

$$F_n\colon H^{n+1} \times [0,1] \multimap H^{n+1} \text{ by } F_n(x,s) = Lx + (1-s)P_n(K(x)) + sP_{n+1}(K(x)).$$

Denote by π_n the flow induced by f_n and by ξ_n the family of flows induced by F_n. The following is the basis of the definition of our index.

Proposition 3.7 *Let $X \subset \mathbb{H}$ be an isolating neighbourhood for π. There exists $n_0 \in N$ such that if $n \geq n_0$ then $X_n = X \cap H^n$ is an isolating neighbourhood for π_n and ξ_n.*

Proof Define a family of L-vector fields $h\colon \mathbb{H} \times [0,1] \multimap \mathbb{H}$ by

$$h(x,s) = Lx + (1+n)(1-ns)P_{n+1}(F(x)) + n[(n+1)s - 1]P_n(F(x))$$

for

$$\frac{1}{n+1} < s \leq \frac{1}{n}$$

and $h(x,0) = f(x)$. One checks h generates a family $\eta\colon \mathbb{H} \times \mathbb{R} \times [0,1] \multimap \mathbb{H}$ of L-flows. By Proposition 3.4, the graph S of the map $[0,1] \ni s \mapsto \mathrm{Inv}(X, \eta_s)$ is compact in $[0,1] \times X$ and $S \cap (\{0\} \times X) \subset \{0\} \times \mathrm{int}\, X$. Therefore, for some $s_0 > 0$, we have $S \cap ([0,s_0] \times X) \subset [0,s_0] \times \mathrm{int}\, X$; in other words, for $0 \leq s \leq s_0$, $\mathrm{Inv}(X, \eta_s) \subset \mathrm{int}\, X$. One takes $n_0 > 1/s_0$. □

The invariant set $S_n = \mathrm{Inv}(X_n, \pi_n)$ admits an index pair(Y_n, Z_n) and its Conley index is $H^*(Y_n, Z_n)$.

Denote $\nu(k) = \dim H_{k+1}^{+}$, where $H_k^{+} = H_k \cap H^{+}$.

One proves (by continuation) that there exists an isomorphism on cohomology

$$c^*\colon H^*(Y_{n+1}/Z_{n+1}) \to H^*(S^{\nu(n)}(Y_n/Z_n)),$$

where S denotes the topological suspension construction. Thus we have an isomorphism

$$\gamma_n\colon H^{k+\nu(n)}(Y_{n+1}, Z_{n+1}) \to H^k(Y_n, Z_n).$$

Define $\rho\colon N \cup \{0\} \to N \cup \{0\}$, $\rho(0) = 0$ and $\rho(n) = \sum_{i=0}^{n-1} \nu(i)$.

Definition 3.8 $CH^q(X) = \varprojlim\{H^{q+\rho(n)}(Y_n, Z_n), \widetilde{\gamma_n}\}$, where $\widetilde{\gamma_n}$ are compositions of the above homomorphisms.

The following properties are proved anlogously to the single-valued versions (using Theorem 2.9).

Proposition 3.9 *Let X be an isolating neighbourhood for an L-flow φ generated by an L-vector field f. If $CH^*(X, \varphi) \neq \{0\}$, then* $\mathrm{Inv}(X, \varphi) \neq \emptyset$. *In particular there is a bounded solution (lying in X) to problem $\dot{x} \in f(x)$.*

Proposition 3.10 (continuation) *Let Λ be a compact, connected and locally contractible metric space. Assume that $\eta : H \times \mathbb{R} \times \Lambda \multimap H$ is a family of L-flows generated by a family of L-vector fields $f : H \times \Lambda \multimap H$. Let X be an isolating neighborhood for the flow $\eta(\cdot, \lambda)$ for some $\lambda \in \Lambda$. Then there exists a compact neighborhood $C \subset \Lambda$ of λ such that $CH^*(X, \eta(\cdot, \mu)) = CH^*(X, \eta(\cdot, \nu))$ for all $\mu, \nu \in C$.*

Proposition 3.11 *Let X, X' be two isolating neighborhoods for an L-flow φ generated by an L-vector field f. Assume that $X \subset X'$ and $\mathrm{Inv}(X', \varphi) \subset \mathrm{int}\, X$. Then $CH^*(X, \varphi) = CH^*(X', \varphi)$.*

4 Clarke's generalized gradient

Let E be a Banach space, $U \subset E$ open , $f \colon U \to \mathbb{R}$ locally Lipschitz function. Let $x \in U$ and $u \in E$; for small $h > 0$ and $y \in U$ from a neighborhood of x, $h^{-1}[f(y+hu) - f(y)] \leq L_x\|u\|$ where L_x is the Lipschitz constant of f around x. Therefore the *Clarke generalized directional derivative of f at x in the direction of u*

$$f^\circ(x; u) := \limsup_{y \to x,\, h \to 0^+} \frac{f(y + hu) - f(y)}{h}.$$

is well-defined and $f^\circ(x; u) \leq L_x\|u\|$.

Given $x \in U$, the *generalized gradient of f at x* is defined by

$$\partial f(x) := \{p \in E^* \mid \forall u \in E \ \langle p, u\rangle \leq f^\circ(x; u)\}.$$

Clearly $\partial f(x)$ is nonempty, weak*-closed and convex; if $p \in \partial f(x)$, then $\|p\| \leq L_x$. By the Alaoglu theorem, $\partial f(x)$ is weak*-compact.

Let us collect some important properties of the generalized gradient.

Theorem 4.1 *Suppose that $f \colon U \to \mathbb{R}$ is locally Lipschitz and let $x \in U$. Then:*

(i) $\partial(-f)(x) = -\partial f(x)$; if $g \colon U \to \mathbb{R}$ is continuously Fréchet differentiable, then $\partial(f + g)(x) = \partial f(x) + \nabla g(x)$ (i.e. $\partial(f + g)(x) = \{p + \nabla g(x) \mid p \in \partial f(x)\}$, where ∇g stands for the Fréchet derivative of g at x);

(ii) if $\mathbb{F}$ is a Banach space and $A \colon \mathbb{F} \to \mathbb{E}$ is a bounded linear operator, then $f \circ A \colon A^{-1}(U) \to \mathbb{R}$ is locally Lipschitz and $\partial(f \circ A)(x) \subset A^*(\partial f(x))$ where $A^* \colon \mathbb{E}^* \to \mathbb{F}^*$ is the adjoint of A;

(iii) if f has an extremum at x, then $0 \in \partial f(x)$;

(iv) (Mean value theorem) if $y \in U$ and $[x, y] := \{(1-\lambda)x+\lambda y \mid \lambda \in [0, 1]\} \subset U$, then there is $\lambda \in (0, 1)$ and $p \in \partial f((1 - \lambda)x + \lambda y)$ such that

$$f(y) - f(x) = \langle p, y - x\rangle.$$

For the proofs see e.g. [5].

5 Hamiltonian systems

Let $G\colon \mathbb{R}\times\mathbb{R}^{2N}\to\mathbb{R}$ be 2π-periodic with respect to the first variable and locally Lipschitz with respect to the second one. We consider the Hamiltonian differential inclusion

$$\dot z \in J\partial G(t,z) \tag{5.1}$$

where

$$J=\begin{bmatrix}0 & -I_N\\ I_N & 0\end{bmatrix}$$

(I_N stands for the unit $(N\times N)$ matrix) is the standard symplectic matrix and $\partial G(t,z)$ denotes the Clarke generalized gradient with respect to $z\in\mathbb{R}^{2N}$. We shall look for nontrivial 2π-periodic solutions, i.e. 2π-periodic absolutely continuous functions $z\colon\mathbb{R}\to\mathbb{R}^{2N}$ such that the inclusion is satisfied for almost all $t\in\mathbb{R}$.

We shall consider the corresponding action functional on the fractional Sobolev space $\mathbb{H}:=H^{1/2}(S^1,\mathbb{R}^{2N})$ (here $S^1:=\mathbb{R}/2\pi Z$ is the circle parameterized over $[0,2\pi]$), and study its critical points.

Recall that $u\in\mathbb{H}$ if and only $u\in L^2(S^1,\mathbb{R}^{2N})$ (i.e. $u\colon\mathbb{R}\to\mathbb{R}^{2N}$ is 2π-periodic and locally square integrable) such that

$$\sum_{k\in Z}|k||\hat u_k|^2<\infty$$

where

$$\hat u_k:=(2\pi)^{-1}\int_0^{2\pi}e^{-ikt}u(t)\,dt\in C^{2N},\quad k\in Z,$$

is the k-th Fourier coefficient of u:

$$u(t)=\sum_{k\in Z}e^{ikt}\hat u_k,\quad \hat u_{-k}=\overline{\hat u_k},\quad k\in Z.$$

Then $\mathbb{H}$ is a real Hilbert space with the inner product:

$$\langle u,v\rangle_{\mathbb{H}}=2\pi\hat u_0\cdot\hat v_0+2\pi\sum_{k\in Z\setminus\{0\}}|k|\hat u_k\cdot\hat v_k,$$

where $\cdot$ is the standard Hermitian product in C^{2N}.

For any $a\in\mathbb{R}$, $e^{aJ}=\cos a\cdot I_{2N}+\sin a\cdot J$. Therefore, for $u\in L^2(S^1,\mathbb{R}^{2N})$, we may write

$$u(t)=\sum_{k\in Z}e^{ktJ}u_k,$$

where $u_k:=(2\pi)^{-1}\int_0^{2\pi}e^{-ktJ}u(t)\,dt$ for $k\in Z$. Then

$$u_k+u_{-k}=2\mathrm{Re}\hat u_k,\ J(u_k-u_{-k})=-2\mathrm{Im}\hat u_k.$$

Hence $u \in \mathbb{H}$ if and only if $u \in L^2(S^1, \mathbb{R}^{2N})$ and

$$\sum_{k \in Z} |k||u_k|^2 < \infty.$$

Using this notation we see that, for $u, v \in \mathbb{H}$, $v(t) := \sum_{k \in Z} e^{ktJ} v_k$,

$$\langle u, v \rangle_{\mathbb{H}} = 2\pi u_0 \cdot v_0 + 2\pi \sum_{k \in Z^*} |k| u_k \cdot v_k$$

Consider a map $L \colon \mathbb{H} \to \mathbb{H}$ given by

$$Lu(t) := \sum_{k \in Z} (\operatorname{sgn} k) e^{ktJ} u_k$$

for $u(t) = \sum_{k \in Z} e^{ktJ} u_k$, i.e. $(Lu)_k = (\operatorname{sgn} k) u_k$ for all $k \in Z$ $(\operatorname{sgn} 0 := 0)$.

We suppose that:

- (G_1) for all $u \in \mathbb{R}^{2N}$, $G(\cdot, u) \colon \mathbb{R} \to \mathbb{R}$ is measurable and 2π-periodic; $G(\cdot, 0) \in L^1_{loc}$;
- (G_2) there exists $k \in L^q([0, 2\pi], \mathbb{R})$ such that, for almost all $t \in [0, 2\pi]$ and all $u, v \in \mathbb{R}^{2N}$, $|G(t, u) - G(t, v)| \leq k(t)|u - v|$, or
- $(G_2)'$ for almost all $t \in \mathbb{R}$, $G(t, \cdot) \colon \mathbb{R}^{2N} \to \mathbb{R}$ is locally Lipschitz and there are $\alpha \in L^q([0, 2\pi], \mathbb{R})$ and $\beta \geq 0$ such that, for almost all $t \in [0, 2\pi]$ and all $u \in \mathbb{R}^{2N}$,

$$\sup_{y \in \partial G(t,u)} |y| \leq \alpha(t) + \beta |u|^{p-1}.$$

Finally we define $\psi \colon \mathbb{H} \to \mathbb{R}$ by the formula

$$\psi(u) := -\int_0^{2\pi} G(t, u(t))\, dt, \quad u \in \mathbb{H}.$$

In order to obtain the existence of periodic solutions we shall study a functional $\Phi \colon \mathbb{H} \to \mathbb{R}$ given by

$$\Phi(u) := \frac{1}{2} \langle Lu, u \rangle_{\mathbb{H}} + \psi(u), \quad u \in \mathbb{H}.$$

Using the compact embedding of $\mathbb{H}$ into L^2 and special properties of the Clarke gradient for integral functional we prove the following:

Proposition 5.1 [9] *The map $\partial\Phi = L + \partial\psi \colon \mathbb{H} \multimap \mathbb{H}$ is an L-vector field.*

Proposition 5.2 *Suppose that in $z \in \mathbb{H}$ is a critical point of Φ, (i.e. $0 \in \partial\Phi(z)$), then $z \in H^1(S^1, \mathbb{R}^{2N}) \subset \mathbb{H}$ and z is a solution to (5.1).*

The following crucial abstract result is not as straightforward as in the smooth case. The reason is that we do not know that an invariant set of a multivalued flow generated by a (generalized) gradient vector field has to contain critical points as it is true in a smooth case.

Theorem 5.3 [9] *Let φ be the L-flow generated by the L-vector field $\partial\Phi$. Assume that N is an isolating neighbourhood for φ and $CH^*(N,\varphi) \neq 0$. Then the set of critical points $K(\Phi) \cap N \neq \emptyset$.*

In the proof we use the following two ingredients.

Theorem 5.4 *For any $\varepsilon > 0$, there exists a locally Lipschitz L-field $W\colon \mathbb{H} \setminus K(\Phi) \to \mathbb{H}$ of the form $W(y) = Ly + V(y)$ where $V\colon \mathbb{H} \setminus K(\Phi) \to \mathbb{H}$ is completely continuous with sublinear growth such that, for each $y \in \mathbb{H}$ and $q \in \partial\Phi(y)$,*

$$\langle W(y), q\rangle_{\mathbb{H}} \geq \frac{1}{2}\delta(y) > 0,$$

where δ is a certain continuous function. Moreover $V(y) \in \overline{\mathrm{conv}}\, \partial\psi(B_\varepsilon(y))$ for every $y \in \mathbb{H} \setminus K(\Phi)$.

Proof For any $u \in \mathbb{H}$, define

$$\|\partial\Phi(u)\| = \inf_{v \in \partial\Phi(u)} \Phi^\circ(u; v).$$

Define a function $\delta\colon \mathbb{H} \to \mathbb{R}$ by the formula

$$\delta(x) := \inf_{u \in \mathbb{H}} (\|\partial\Phi(u)\| + \|u - x\|_{\mathbb{H}}), \ x \in \mathbb{H}.$$

Observe that for $u \in \mathbb{H} \setminus K(\Phi)$ we have $\|\partial\Phi(u)\| \geq \delta(u) > 0$. Hence there is $v_u \in \partial\psi(u)$ such that

$$\inf_{q \in \partial\Phi(u)} \langle q, Lu + v_u\rangle_{\mathbb{H}} > \frac{1}{2}\delta(u).$$

Observe now that

$$\inf_{q \in \partial\Phi(u)} \langle q, Lu + v_u\rangle_{\mathbb{H}} = -\Phi^\circ(u; -Lu - v_u).$$

The function

$$\mathbb{H} \ni y \mapsto \inf_{q \in \partial\Phi(y)} \langle q, Ly + v_u\rangle_{\mathbb{H}} - \frac{1}{2}\delta(y)$$

is lower semicontinuous and takes a positive value for $y = u$. Hence there is a open neighborhood $N_u \subset B_\varepsilon(u)$ of u such that, for all $y \in N_u$,

$$\inf_{q \in \partial\Phi(y)} \langle q, Ly + v_u\rangle_{\mathbb{H}} > \frac{1}{2}\delta(y).$$

Consider a locally finite partition of unity $\{\lambda_s\}_{s\in S}$ consisting of locally Lipschitz functions with supports $\{\operatorname{supp}\lambda_s\}$ refining the cover $\{N_u\}_{u\in\mathbb{H}\setminus K(\Phi)}$ of $\mathbb{H}\setminus K(\Phi)$, i.e. for any $s\in S$, there is $u_s\in\mathbb{H}$ such that $\operatorname{supp}\lambda_s\subset N_{u_s}$. For each $s\in S$, let $v_s:=v_{u_s}$ and define

$$V(y):=\sum_{s\in S}\lambda_s(y)v_s,\ \ y\in\mathbb{H}\setminus K(\Phi).$$

It is clear that V is well-defined, locally Lipschitz and maps bounded sets in $\mathbb{H}\setminus K(\Phi)$ into compact ones. Moreover, V has sublinear growth since so does $\partial\psi$.

Let $y\in\mathbb{H}\setminus K(\Phi)$ and let $S_y:=\{s\in S\mid\lambda_s(y)\neq 0\}$. If $s\in S_y$, then $y\in N_{u_s}\subset B_\varepsilon(u_s)$; hence $u_s\in B_\varepsilon(y)$ and $v_s\in\partial\psi(B_\varepsilon(y))$. Therefore $V(y)\in\overline{\operatorname{conv}}\,\partial\psi(B_\varepsilon(y))$ and

$$\inf_{q\in\partial\Phi(y)}\langle q,Ly+v_s\rangle_{\mathbb{H}}>\frac{1}{2}\delta(y).$$

Hence, for any $q\in\partial\Phi(y)$

$$\langle q,Ly+V(y)\rangle_{\mathbb{H}}=\sum_{s\in S_y}\lambda_s(y)\langle q,Ly+v_s\rangle_{\mathbb{H}}\geq\frac{1}{2}\delta(y).$$

Putting $W(y):=Ly+V(y)$, $y\in\mathbb{H}\setminus K(\Phi)$ we complete the proof. □

Theorem 5.5 *Suppose that y belongs to the ω-limit set $\omega(x)$ of the point $x\in\mathbb{H}\setminus K(\Phi)$ with respect to the local dynamical system η generated by W. Then $y\in K(\Phi)$.*

Proof Recall that, by definition $y\in\omega(x)$ if and only if $y=\lim_{t\to t^+(x)}\eta(x,t)$ where $\dot\eta(x,t)=W(\eta(x,t))$ for $t\in J_x:=(t^-(x),t^+(x))$. As above we see that Φ increases along η. We show that Φ is constant on $\omega(x)$. Indeed, let $z\in\omega(x)$. Hence there are sequences $t_n\to t^+(x)$, $s_n\to t^+(x)$ such that $\eta(x,t_n)\to y$ and $\eta(x,s_n)\to z$. We may assume that $\cdots<t_n<s_n<t_{n+1}<s_{n+1}<\ldots t^+(x)$. Hence $\cdots<\Phi(\eta(x,t_n))<\Phi(\eta(x,s_n))<\Phi(\eta(x,t_{n+1}))<\Phi(\eta(x,s_{n+1}))<\ldots$. By continuity, $\Phi(y)=\Phi(z)$.

Suppose to the contrary that $y\notin K(\Phi)$, i.e. $\delta(y)>0$, and consider the trajectory $\eta(y,\cdot)\colon J_y\to\mathbb{H}$. It is easy to see that $\{\eta(y,t)\mid t\in J_y\}\subset\omega(x)$. But then (at least locally) Φ is strictly decreasing along $\eta(y,\cdot)$and we have a contradiction. □

Proof of Theorem 5.3 Suppose to the contrary that $K(\Phi)\cap N=\emptyset$ and let $\varepsilon_0>0$ be sufficiently small. Then by the continuation property (Proposition 3.10), $CH(N,\varphi_\varepsilon)=CH(N,\varphi)\neq 0$ for any $0<\varepsilon\leq\varepsilon_0$, where φ_ε is actually a flow generated by the field $f_\varepsilon:=L+\overline{\operatorname{conv}}\,\partial\psi(B_\varepsilon(\cdot))$.

Fix $0<\varepsilon\leq\varepsilon_0$ and let $\lambda\colon H\to[0,1]$ be given by

$$\lambda(x)=\frac{d(x,K(\Phi))}{d(x,N)+d(x,K(\Phi))},\quad x\in H.$$

Then $\lambda|_N \equiv 1$ and $\lambda|_{K(\Phi)} \equiv 0$. Take a map $V\colon H \setminus K(\Phi) \to H$ given by Proposition 5.4 and let $\widetilde{V}(x) = \lambda(x)V(x)$ for $x \in H \setminus K(\Phi)$ and $\widetilde{V}(x) = 0$ for $x \in K(\Phi)$. Then $\widetilde{V}\colon H \to H$ is locally Lipschitz, completely continuous and has sublinear growth. Let $\widetilde{\eta}$ be an LS-flow generated by $\widetilde{W} := L + \widetilde{V}$.

Observe that $\widetilde{W}$ is a selection of f_ϵ. Thus a linear homotopy $h\colon H \times [0,1] \multimap H$, where

$$h(x,t) = tf_\varepsilon(x) + (1-t)\widetilde{W}(x), \quad t \in [0,1],\ x \in H,$$

generates a family of LS-flows $\theta\colon H \times \mathbb{R} \times [0,1] \multimap H$. For each $t \in [0,1]$ and $x \in H$, $h(x,t) \subset f_\varepsilon(x)$. Thus, for all $t \in [0,1]$, N is an isolating neighborhood for $\theta(\cdot,t)$. Therefore

$$CH(N,\widetilde{\eta}) = CH(N,\theta(\cdot,1)) = CH(N,\theta(\cdot,0)) = CH(N,\varphi_\varepsilon) \neq 0.$$

By Proposition 3.9, the invariant part $\mathrm{Inv}(N,\widetilde{\eta})$ is nonempty. Hence, for $x \in \mathrm{Inv}(N,\widetilde{\eta})$, $\emptyset \neq \omega(x) \subset N$ since $\mathrm{Inv}(N,\widetilde{\eta})$ is compact. By Theorem 5.5 we obtain that there is $y \in K(\Phi) \cap N$. This contradiction proves the theorem. □

Now we apply the index to asymptotically linear Hamiltonians.

Given a symmetric $2N \times 2N$-matrix A with real (constant) coefficients, consider the following Hamiltonian system

$$\dot{z} = JAz.$$

For a symmetric (real) matrix B, let $M^{\pm}(B)$ and $M^0(B)$, denote the number (with multiplicity) of positive (resp. negative) eigenvalues of B and the dimension of its kernel, respectively, and define the *generalized Morse index*

$$i^{\pm}(A) := M^{\pm}(-A) + \sum_{k=1}^{\infty}(M^{\pm}(T_k(A)) - 2N),$$

and the *generalized nullity*

$$i^0(A) := M^0(-A) + \sum_{k=1}^{\infty} M^0(T_k(A)).$$

where

$$T_k(A) = \begin{bmatrix} -\frac{1}{k}A & -J \\ J & -\frac{1}{k}A \end{bmatrix}$$

Now let us assume that $G\colon \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ satisfies condition (G_1) with $G(\cdot,0) \equiv 0$ and (G_2) with $k(\cdot) \equiv k \geq 0$ or $(G_2)'$ with $\alpha(\cdot) \equiv \alpha \geq 0$. Moreover let us assume that there are symmetric $2N \times 2N$-matrices A_0 and A_∞ such that:

- (G_3) $\sup_{y \in \partial G(t,u)} |y - A_0 u| = o(|u|)$ as $u \to 0$ uniformly with respect to $t \in [0,2\pi]$;
- (G_4) $\sup_{y \in \partial G(t,u)} |y - A_\infty u| = o(|u|)$ as $|u| \to \infty$ uniformly with respect to $t \in [0,2\pi]$.

The following theorem constitutes a multivalued version of the result due to H. Amann and E. Zehnder [1] (see also [22], [18]).

Theorem 5.6 [9] *Let us consider the Hamiltonian system (5.1) with asymptotically linear G, i.e. satisfy* (G_1)–(G_4)*. Assume that* $i^0(A_0) = i^0(A_\infty) = 0$ *and* $i^+(A_0) \neq i^+(A_\infty)$ *or* $i^-(A_0) \neq i^-(A_\infty)$*. Then the system (5.1) has a nontrivial solution (in addition to the trivial one* $x = 0$*).*

The idea of proof is standard: the assumptions assure that there are two isolating neighborhoods of 0 with different indices. The non-resonance assumption simplifies the proof. However there are many results in the smooth case with resonance, see e.g. [3], [11], [15], [18], [23] and references therein. Some of them can be generalised to our generality. Let us finish with an illustrating example.

Example 5.7 Suppose $\alpha\colon \mathbb{R}_+ \to [0,1]$ is such that $\alpha|_{[0,1]} \equiv 0$, $\alpha|_{[3,+\infty)} \equiv 2$ and $\alpha(t) = t-1$ for $1 \leq t \leq 3$. Let $\eta \in C^1(\mathbb{R}^2, \mathbb{R})$ be bounded with the bounded $\nabla\eta$, $\eta(0) = 0$ and $|\nabla\eta(x)| = o(|x|)$ as $|x| \to 0$. Suppose that $g\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is bounded, measurable and 2π-periodic with respect to $t \in \mathbb{R}$ and Lipschitz with respect to $x \in \mathbb{R}^2$. Finally suppose that $g(\cdot, 0) \equiv 0$. Let $G\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ be given by

$$G(t,x) := \frac{1+\alpha(|x|)}{5}|x|^2 + \eta(x)g(t,x), \quad t \in \mathbb{R},\ x \in \mathbb{R}^2. \tag{5.2}$$

Then, for $|x| \leq 1$,

$$G(t,x) = \frac{1}{2}\,A_0 x \cdot x + \eta(x)g(t,x)$$

and, for $|x| \geq 2$,

$$G(t,x) = \frac{1}{2}\,A_\infty x \cdot x + \eta(x)g(t,x)$$

where $A_0 := \frac{2}{5}I_2$ and $A_\infty := \frac{6}{5}I_2$. It is clear that G satisfies (G_1) and (G_2) and, for any $t \in \mathbb{R}$,

$$\partial G(t,x) = \begin{cases} A_0 x + g(t,x)\nabla\eta(x) + \eta(x)\partial g(t,x) & \text{for } |x| < 1; \\ A_\infty x + g(t,x)\nabla\eta(x) + \eta(x)\partial g(t,x) & \text{for } |x| > 2. \end{cases}$$

Thus conditions (G_3) and (G_4) are satisfied, too. One easily verifies that $i^0(A_0) = i^0(A_\infty) = 0$, $i^-(A_0) = 2$ and $i^-(A_\infty) = 4$; therefore the last theorem provides a nontrivial periodic solution to (5.1) with G given by (5.2).

Notice that if we assume in the above example some symmetry, we obtain more nontrivial solutions. For instance, assume additionally, that the functions η hand g are even with respect to x, i.e. $\eta(-x) = \eta(x)$ and $g(t,-x) = g(t,x)$. Then one can easily verify that the induced multivalued flow in the Hilbert space is Z_2-equivariant and we obtain at least two non-trivial solutions. The similar schedule can be used with more involved symmetries.

References

[1] Amann, H., Zehnder, E.: *Periodic solutions of asymptotically linear Hamiltonian systems*. Manuscripta Math. **32** (1980), 149–189.

[2] Bobylev, N. A., Bulatov, A. V., Kuznetsov, Yu. O.: *An approximation scheme for defining the Conley index of isolated critical points*. Diff. Equ. **40** (2004), 1539–1544.

[3] Chang, K. C., Liu, J. Q., Liu, M. J.: *Nontrivial periodic solutions for strong resonance Hamiltonian systems*. Ann. Inst. Henri Poincare **14** (1997), 103–117.

[4] Clarke, F. H.: *Periodic solutions to Hamiltonian inclusions*. J. Diff. Eq. **40** (1981), 1–6.

[5] Clarke, F. H.: Optimization and Nonsmooth Analysis. *Wiley*, New York, 1983.

[6] Conley, C.: Isolated Invariant Sets and the Morse Index. CBMS **38** *Amer. Math. Society*, Providence, Rhode Island, 1978.

[7] Dzedzej, Z.: On the Conley index in Hilbert spaces – a multivalued case. *Banach Center Publ.* **77** Warszawa, 2007, 61–68.

[8] Dzedzej, Z., Gabor, G.: *On homotopy Conley index for multivalued flows in Hilbert spaces*. to appear.

[9] Dzedzej, Z., Kryszewski, W.: *Conley type index applied to Hamiltonian inclusions*. J. Math. Anal. Appl. **347** (2008), 96–112.

[10] Emelyanov, S. V., Korovin, S. K., Bobylev, N. A., Bulatov, A. V.: Homotopy of Extremal Problems. *W. de Gruyter*, Berlin, 2007.

[11] Fei, G. H.: *Maslov-type index and periodic solution of asymptotically linear Hamiltonian systems which are resonant at infinity*. J. Differential Equations **121** (1995), 121–133.

[12] Gabor, G.: *Homotopy index for multivalued flows on sleek sets*. Set-valued Analysis **13** (2005), 125–149.

[13] Gęba, K., Izydorek, M., Pruszko, A.: *The Conley index in Hilbert spaces and its applications*. Studia Math. **134** (1999), 217–233.

[14] Górniewicz, L.: Topological Fixed Point Theory of Multivalued Mappings. *Kluwer*, Dordrecht–Boston–London, 1999.

[15] Han, Z. Q.: *Computations of cohomology groups and nontrivial periodic solutions of Hamiltonian systems*. J. Math. Anal. Appl. **330** (2007), 259–275.

[16] Izydorek, M.: *A cohomological index in Hilbert spaces and applications to strongly indefinite problems*. J. Differential Equations **170** (2001), 22–50.

[17] Izydorek, M., Rybakowski, K.: *On the Conley index in the absence of uniqueness*. Fund. Math. **171** (2002), 31–52.

[18] Kryszewski, W., Szulkin, A.: *An infinite dimensional Morse theory with applications*. Trans. Amer. Math. Soc. **349** (1997), 3181–3234.

[19] Kunze, M., Küpper, T., Li, Y.: *On Conley index theory for non-smooth dynamical systems*. Differential Integral Equations **13** (2000), 479–502.

[20] Mrozek, M.: *A cohomological index of Conley type for multivalued admissible flows*. J. Differential Equations **84** (1990), 15–51.

[21] Rybakowski, K.: *The Homotopy Index and Partial Differential Equations*. *Springer*, Berlin, 1987.

[22] Szulkin, A.: *Cohomology and Morse theory for strongly indefinite functionals*. Math. Z. **209** (1992), 375–418.

[23] Szulkin, A., Zhou, W.: *Infinite dimensional cohomology groups and periodic solutions of asymptotically linear Hamiltonian systems*. J. Differential Equations **174** (2001), 369–391.