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Convergence Theorems for a Finite Family of Nonexpansive and Asymptotically Nonexpansive Mappings*

Kittipong SITTHIKUL [1], Satit SAEJUNG [2] **

Department of Mathematics, Khon Kaen University,
Khon Kaen 40002, Thailand
[1] *e-mail: sittikul_n@hotmail.com*
[2] *e-mail: saejung@kku.ac.th*

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Abstract

In this paper, weak and strong convergence of finite step iteration sequences to a common fixed point for a pair of a finite family of nonexpansive mappings and a finite family of asymptotically nonexpansive mappings in a nonempty closed convex subset of uniformly convex Banach spaces are presented.

Key words: Nonexpansive mapping, asymptotically nonexpansive mapping, common fixed point, finite-step iterative sequence.

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1 Introduction

The class of asymptotically nonexpansive mappings which is an important generalization of that of nonexpansive mappings was introduced by Goebel and Kirk [6]. Iteration processes for nonexpansive and asymptotically nonexpansive mappings in Banach spaces including Mann [11] and Ishikawa [8] iteration processes have been studied extensively by many authors (see [2, 7, 14, 15, 16, 17]).

Recently, Xu and Noor [19] introduced and studied a three-step scheme to approximate fixed points of asymptotically nonexpansive mappings in Banach space. Cho et al. [3] extended the work of Xu and Noor [19] to the three-step

*Supported by the Faculty of Science, Khon Kaen University.
**Correspondence should be addressed to Satit Saejung, saejung@kku.ac.th.

iterative scheme with errors in a Banach space and gave weak and strong convergence theorems for asymptotically nonexpansive mappings. Chidume and Ali [2] considered the multi-step scheme for a finite family of asymptotically nonexpansive mappings and gave weak convergence theorems for this scheme in a uniformly convex Banach space whose the dual space has the Kadec–Klee property. They also proved a strong convergence theorem under some appropriate conditions on a finite family of asymptotically nonexpansive mappings. Liu et al. (see [9] and [10]) established a new method with respect to a pair of nonexpansive and asymptotically nonexpansive mappings. The results in [9] and [10] generalize, improve and unify many known results due to many authors. Moreover, they also gave an example to demonstrate that their results are substantial generalizations and many previous known results are not applicable in this case.

Inspired by the above works, in this paper, a multi-step iteration scheme for a finite family of nonexpansive and asymptotically nonexpansive mappings is introduced and strong and weak convergence theorems of this scheme to common fixed point of nonexpansive and asymptotically nonexpansive mappings are proved. The weak convergence theorem is proved in a uniformly convex Banach space whose dual has the Kadec–Klee property. It is worth mentioning that there are uniformly convex Banach spaces, which have neither a Fréchet differentiable norm nor Opial property; however, their dual does have the Kadec–Klee property (see [5, Example 3.1]). Hence our results are different from [9] and [10] and the proofs are of independent interest.

2 Preliminaries

Let K be a nonempty subset of a real Banach space E and $T\colon K \to K$ be a mapping with the fixed point set $F(T)$, i.e., $F(T) = \{x \in K\colon x = Tx\}$. In this paper, we write $x_n \to x$ (resp. $x_n \rightharpoonup x$) if x_n converges strongly (resp. weakly) to x.

Definition 1 A mapping $T\colon K \to K$ is said to be

1. *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all $x, y \in K$ and $n \ge 1$;

2. *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$;

3. *Lipschitzian (with a Lipschitz constant L)* if $\|Tx - Ty\| \le L\|x - y\|$ for all $x, y \in K$;

4. *demi-closed at a point $p \in K$* if whenever $\{x_n\}$ is a sequence in K which converges weakly to a point $x \in K$ and $\{Tx_n\}$ converges strongly to p, it follows that $Tx = p$.

Definition 2 [4] A norm on a Banach space E is *uniformly convex* (or simply, E is uniformly convex) if for all $\{x_n\}, \{y_n\} \subset \{z \in E\colon \|z\| = 1\}$ such that $\left\|\frac{x_n+y_n}{2}\right\| \to 1$, we have $\|x_n - y_n\| \to 0$.

Let K be a nonempty subset of a Banach space E. Let $S_1, S_2, \ldots, S_N\colon K \to K$ be N nonexpansive mappings, $T_1, T_2, \ldots, T_N\colon K \to K$ be N asymptotically nonexpansive mappings. Then the sequence $\{x_n\}$ defined by

$$\left.\begin{aligned}
x_1 &\in K,\\
x_n^{(0)} &= x_n,\\
x_n^{(1)} &= a_n^{(1)} T_1^n x_n^{(0)} + (1 - a_n^{(1)}) S_1 x_n,\\
x_n^{(2)} &= a_n^{(2)} T_2^n x_n^{(1)} + (1 - a_n^{(2)}) S_2 x_n,\\
&\vdots\\
x_n^{(N-1)} &= a_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1 - a_n^{(N-1)}) S_{N-1} x_n,\\
x_n^{(N)} &= a_n^{(N)} T_N^n x_n^{(N-1)} + (1 - a_n^{(N)}) S_N x_n,\\
x_{n+1} &= x_n^{(N)}, \quad n \geq 1,
\end{aligned}\right\} \tag{1}$$

where $\{a_n^{(i)}\}_{n=1}^{\infty} \subset [0, 1]$, $i = 1, 2, \ldots, N$. An example of such iterations can be found in [9] and [10].

The purpose of this paper is to study the weak and strong convergences of finite-step iteration sequence $\{x_n\}$ defined by (1) to a common fixed point of a finite family of nonexpansive mappings and a finite family of asymptotically nonexpansive mappings in a uniformly convex Banach space.

The following lemmas are our main tool for proving the results.

Lemma 1 ([7]) *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. If $T\colon K \to K$ is an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero.*

Lemma 2 *Let E be a uniformly convex Banach space, $\{x_n\}$ and $\{y_n\}$ be sequences in E. Suppose that there is $\delta > 0$ such that $\delta \leq t_n \leq 1 - \delta$ for all $n \in \mathbb{N}$. If $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = a$ for some $a \geq 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Moreover, $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = a$.*

Proof The first assertion follows from [15]. It suffices to prove that

$$\liminf_{n\to\infty} \|x_n\| \geq a.$$

In fact, this follows since

$$a = \lim_{n\to\infty} \|t_n x_n + (1 - t_n) y_n\| = \lim_{n\to\infty} \|x_n + (1 - t_n)(y_n - x_n)\|.$$

This finishes the proof. □

Lemma 3 ([13]) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative numbers satisfying the inequality $a_{n+1} \leq (1 + b_n)a_n$, for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 4 ([5]) *Let E be a reflexive Banach space such that its dual E^* has the Kadec–Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p, q \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the set of all weak cluster points of the sequence $\{x_n\}$. Suppose that $\lim_{n\to\infty} \|tx_n + (1 - t)p - q\|$ exists for all $t \in [0, 1]$. Then $p = q$.*

Lemma 5 ([5]) *Let K be a convex subset of a uniformly convex Banach space E. Then there exists a strictly continuous convex function $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for each Lipschitzian mapping $T\colon K \to K$ with a Lipschitz constant L,*

$$\|tTx + (1 - t)Ty - T(tx + (1 - t)y)\| \leq L\phi^{-1}(\|x - y\| - \frac{1}{L}\|Tx - Ty\|)$$

for all $x, y \in K$ and all $0 < t < 1$.

Proposition 1 ([20]) *Let K be a nonempty subset of a Banach space E and $T_1, T_2 \ldots, T_N\colon K \to K$ be N asymptotically nonexpansive mappings. Then there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and*

$$\|T_i^n x - T_i^n y\| \leq k_n\|x - y\| \tag{2}$$

for all $x, y \in K$, $n \geq 1$ and $i = 1, 2, \ldots, N$.

From now on, we will assume that N asymptotically nonexpansive mappings $T_1, T_2, \ldots, T_N\colon K \to K$ share the same sequence $\{k_n\} \subset [1, \infty)$ as mentioned in the preceding proposition.

3 Technical Lemmas

Lemma 6 *Let K be a nonempty convex subset of a real Banach space E. Let $S_1, S_2, \ldots, S_N\colon K \to K$ be nonexpansive mappings, $T_1, T_2, \ldots, T_N\colon K \to K$ be asymptotically nonexpansive mappings with the sequence $\{k_n\}$ and suppose that $F = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \varnothing$. If*

$$\sum_{n=1}^{\infty}(k_n - 1) < \infty, \tag{3}$$

then $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F$, where $\{x_n\}$ is defined by the iterative scheme (1).

Proof Let $q \in F$. It follows from (1) that

$$\begin{aligned}\|x_n^{(1)} - q\| &\leq a_n^{(1)}\|T_1^n x_n - q\| + (1 - a_n^{(1)})\|S_1 x_n - q\| \\ &\leq a_n^{(1)} k_n\|x_n - q\| + (1 - a_n^{(1)})\|x_n - q\| \\ &\leq a_n^{(1)} k_n\|x_n - q\| + (1 - a_n^{(1)})k_n\|x_n - q\| \\ &= k_n\|x_n - q\| \end{aligned} \tag{4}$$

and from (4), we have

$$\begin{aligned}\|x_n^{(2)} - q\| &\le a_n^{(2)}\|T_2^n x_n^{(1)} - q\| + (1 - a_n^{(2)})\|S_2 x_n - q\| \\ &\le a_n^{(2)} k_n \|x_n^{(1)} - q\| + (1 - a_n^{(2)})\|x_n - q\| \\ &\le a_n^{(2)} k_n^2 \|x_n - q\| + (1 - a_n^{(2)}) k_n^2 \|x_n - q\| \\ &= k_n^2 \|x_n - q\|. \end{aligned} \tag{5}$$

Continuing the above process, we get

$$\|x_n^{(i)} - q\| \le k_n^i \|x_n - q\| \quad \text{for all } n \ge 1,\ i = 1, 2, \ldots, N. \tag{6}$$

In particular,

$$\|x_{n+1} - q\| = \|x_n^{(N)} - q\| \le k_n^N \|x_n - q\| = (1 + (k_n^N - 1))\|x_n - q\|.$$

Notice that (3) holds (if and) only if

$$\sum_{n=1}^{\infty} (k_n^N - 1) < \infty. \tag{7}$$

By Lemma 3, we have $\lim_{n\to\infty} \|x_n - q\|$ exists. This completes the proof. □

Lemma 7 *Under the assumptions of Lemma 6 and suppose that there is $\delta > 0$ such that*

$$\delta \le a_n^{(i)} \le 1 - \delta \quad \textit{for all } n \ge 1, \quad i = 1, 2, \ldots, N. \tag{8}$$

If $\{x_n\}$ is defined by the iterative scheme (1), then

$$\lim_{n\to\infty} \|S_i x_n - T_i^n x_n^{(i-1)}\| = 0 \quad \textit{for all } i = 1, 2, \ldots, N. \tag{9}$$

Proof Let $q \in F$. By Lemma 6, we have

$$d = \lim_{n\to\infty} \|x_n - q\| \text{ exists}. \tag{10}$$

It follows from (6), (10) and $\lim_{n\to\infty} k_n = 1$ that

$$\limsup_{n\to\infty} \|x_n^{(N-1)} - q\| \le d, \tag{11}$$

and so

$$\limsup_{n\to\infty} \|T_N^n x_n^{(N-1)} - q\| \le d.$$

Also,

$$\limsup_{n\to\infty} \|S_N x_n - q\| \le d.$$

Further, from (10) and (1) we have

$$d = \lim_{n\to\infty} \|x_n^{(N)} - q\| = \lim_{n\to\infty} \|a_n^{(N)}(T_N^n x_n^{(N-1)} - q) + (1 - a_n^{(N)})(S_N x_n - q)\|.$$

Then, by Lemma 2, we get

$$\lim_{n\to\infty} \|S_N x_n - T_N^n x_n^{(N-1)}\| = \lim_{n\to\infty} \|(S_N x_n - q) - (T_N^n x_n^{(N-1)} - q)\| = 0,$$

and

$$\lim_{n\to\infty} \|T_N^n x_n^{(N-1)} - q\| = d.$$

Therefore,

$$\begin{aligned} d = \liminf_{n\to\infty} \|T_N^n x_n^{(N-1)} - q\| &\le \liminf_{n\to\infty} k_n \|x_n^{(N-1)} - q\| \\ = \liminf_{n\to\infty} \|x_n^{(N-1)} - q\| &\le \limsup_{n\to\infty} \|x_n^{(N-1)} - q\| \le d. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|x_n^{(N-1)} - q\| = d.$$

It follows from (6), (10) and $\lim_{n\to\infty} k_n = 1$ that

$$\limsup_{n\to\infty} \|x_n^{(N-2)} - q\| \le d, \tag{12}$$

and so

$$\limsup_{n\to\infty} \|T_{N-1}^n x_n^{(N-2)} - q\| \le d.$$

Also,

$$\limsup_{n\to\infty} \|S_{N-1} x_n - q\| \le d.$$

Further, from (10) and (1) we have

$$d = \lim_{n\to\infty} \|x_n^{(N-1)} - q\| = \lim_{n\to\infty} \|a_n^{(N-1)}(T_{N-1}^n x_n^{(N-2)} - q) + (1 - a_n^{(N-1)})(S_{N-1} x_n - q)\|.$$

Applying Lemma 2, we have

$$\lim_{n\to\infty} \|S_{N-1} x_n - T_{N-1}^n x_n^{(N-2)}\| = \lim_{n\to\infty} \|(S_{N-1} x_n - q) - (T_{N-1}^n x_n^{(N-2)} - q)\| = 0.$$

Continuing this in an obvious manner, we get (9) and this completes the proof. □

Lemma 8 *Under the assumptions of Lemma 6 and suppose that (8) holds. If $\{x_n\}$ is defined by the iterative scheme (1) and*

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0 \quad \textit{for all } i = 1, 2, \ldots, N, \tag{13}$$

then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ *for all* $i = 1, 2, \ldots, N$.

Proof By Lemma 7, we have

$$\lim_{n\to\infty} \|S_i x_n - T_i^n x_n^{(i-1)}\| = 0, \quad \text{for all } i = 1, 2, \ldots, N. \tag{14}$$

It follows from (13) that,

$$\lim_{n\to\infty} \|x_n - T_i^n x_n^{(i-1)}\| = 0, \quad \text{for all } i = 1, 2, \ldots, N. \tag{15}$$

Next, from (1) we have

$$\|x_n - x_{n+1}\| \le a_n^{(N)} \|x_n - T_N^n x_n^{(N-1)}\| + (1 - a_n^{(N)}) \|x_n - S_N x_n\|.$$

From (13) and (15), we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{16}$$

Thus, we can estimate, using (1),

$$\begin{aligned} \|x_n - T_i^n x_n\| &\le \|x_n - T_i^n x_n^{(i-1)}\| + \|T_i^n x_n^{(i-1)} - T_i^n x_n\| \\ &\le \|x_n - T_i^n x_n^{(i-1)}\| + k_n \|x_n^{(i-1)} - x_n\| \\ &\le \|x_n - T_i^n x_n^{(i-1)}\| + k_n a_n^{(i-1)} \|T_{i-1}^n x_n^{(i-2)} - x_n\| \\ &\quad + k_n (1 - a_n^{(i-1)}) \|S_{i-1} x_n - x_n\|. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0, \quad \text{for all } i = 1, 2, \ldots, N. \tag{17}$$

It then follows from (16) and (17) that

$$\begin{aligned} \|x_n - T_i x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| \\ &\quad + \|T_i^{n+1} x_n - T_i x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + k_{n+1} \|x_{n+1} - x_n\| \\ &\quad + k_1 \|T_i^n x_n - x_n\| \\ &\le (1 + k_{n+1}) \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + k_1 \|T_i^n x_n - x_n\| \end{aligned}$$

for $i = 1, 2, \ldots, N$. This implies that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \text{for all } i = 1, 2, \ldots, N.$$

This completes the proof. □

Lemma 9 *Under the assumptions of Lemma 6 and suppose that (8) holds and that*

$$\|x - T_i y\| \le \|S_i x - T_i y\| \quad \text{for all } x, y \in K \text{ and } i = 1, 2, \ldots, N. \tag{18}$$

If the sequence $\{x_n\}$ is defined by the iterative scheme (1), then

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0,$$

for all $i = 1, 2, \ldots, N$.

Proof We shall show that

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = 0, \quad \text{for all } i = 1, 2, \dots, N. \tag{19}$$

By Lemma 7, we have

$$\lim_{n\to\infty} \|S_i x_n - T_i^n x_n^{(i-1)}\| = 0, \quad \text{for all } i = 1, 2, \dots, N. \tag{20}$$

It follows from (18) that

$$\lim_{n\to\infty} \|x_n - T_i^n x_n^{(i-1)}\| = 0, \quad \text{for all } i = 1, 2, \dots, N. \tag{21}$$

Thus (19) follows. And Lemma 8 guarantees the second equality. □

4 Strong convergence theorems

A finite family $\{T_1, \dots, T_N\}$ of mappings of K with

$$F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$$

is said to satisfy condition (B) [2] if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0,\infty)$ such that for all $x \in K$

$$\max_{1\le i\le N} \|x - T_i x\| \ge f(d(x,F)),$$

where $d(x,F) = \inf\{\|x-p\|\colon p \in F\}$.

Theorem 1 *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S_1, S_2, \dots, S_N\colon K \to K$ be nonexpansive mappings, $T_1, T_2, \dots, T_N\colon K \to K$ be asymptotically nonexpansive mappings with the sequence $\{k_n\}$ and suppose that $F = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \varnothing$. Suppose that the family $\{S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N\}$ satisfies condition (B) and (3), (8), (18) hold. Then the sequence $\{x_n\}$ defined by (1) converges strongly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$.*

Proof We have

$$\|x_{n+1} - q\| = \|x_n^{(N)} - q\| \le (1 + (k_n^N - 1))\|x_n - q\| \quad \text{for all } q \in F.$$

Consequently,

$$d(x_{n+1}, F) \le (1 + (k_n^N - 1)) d(x_n, F).$$

Applying Lemma 3 to the above inequality, we obtain that $\lim_{n\to\infty} d(x_n, F)$ exists. Also, by Lemma 9,

$$\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \quad \text{for all } i = 1, 2, \dots, N. \tag{22}$$

Since $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ satisfies condition (B), we conclude that

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

We now prove that $\{x_n\}$ is a Cauchy sequence in K. Let $\varepsilon > 0$. Then there exists a positive integer n_0 such that $d(x_{n_0}, F) < \frac{\varepsilon}{4}$. Find $p \in F$ such that $\|x_{n_0} - p\| < \frac{\varepsilon}{4}$. By Lemma 6, we see that $\lim_{n\to\infty} \|x_n - p\|$ exists and so $\{x_n - p\}$ is bounded. Then there is a constant $M > 0$ such that

$$\|x_n - p\| \leq M \quad \text{for all } n \geq 1.$$

We now choose a positive integer n_1 such that

$$\sum_{j=n_1}^{\infty} (k_j^N - 1) < \frac{\varepsilon}{4M}.$$

Moreover, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\| + M(k_n^N - 1).$$

This implies that

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + M(k_{n+m-1}^N - 1) \\ &\leq \|x_{n+m-2} - p\| + M(k_{n+m-2}^N - 1) + M(k_{n+m-1}^N - 1) \\ &\leq \|x_n - p\| + M \sum_{j=n}^{n+m-1} (k_j^N - 1) \end{aligned} \tag{23}$$

for all $n, m \geq 1$. From (23) it follows that, for all $n > n_1$ and $m \geq 1$,

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &\leq 2\|x_{n_1} - p\| + M \sum_{j=n_1}^{n+m-1} (k_j^N - 1) + M \sum_{j=n_1}^{n-1} (k_j^N - 1) \\ &\leq 2\|x_{n_1} - p\| + 2M \sum_{j=n_1}^{n+m-1} (k_j^N - 1) \\ &\leq 2\|x_{n_1} - p\| + 2M \sum_{j=n_1}^{\infty} (k_j^N - 1) \\ &< 2\frac{\varepsilon}{4} + 2M\frac{\varepsilon}{4M} = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in K. In virtue of the completeness of K, we assume that $x_n \to p' \in K$ as $n \to \infty$. By the continuities of S_i and T_i and (22), we have $S_i p' = p' = T_i p'$ for all $i = 1, 2, \ldots, N$, so $p' \in F$. This completes the proof. □

5 Weak convergence theorems

Lemma 10 *Under the assumptions of Lemma 6 and suppose that (8) holds. Let* $\{x_n\}$ *be the sequence defined by (1). Then for all* $u, v \in F$*, the limit* $\lim_{n\to\infty} \|tx_n + (1-t)u - v\|$ *exists for all* $t \in [0,1]$.

Proof Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $\{x_n\} \subset C := B_R(0) \cap K$. Then C is a nonempty closed convex bounded subset of E. Basically, we shall follow the idea of [17]. Let

$$a_n(t) = \|tx_n + (1-t)u - v\|, \quad \text{where } t \in [0,1].$$

Then $a_n(0) = \|u - v\|$, and from Lemma 6, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - v\|$ exists. We now assume that $t \in (0,1)$. Define $U_n \colon C \to C$ by

$$\begin{aligned}
x^{(1)} &= a_n^{(1)} T_1^n x + (1 - a_n^{(1)}) S_1 x, \quad x \in K \\
x^{(2)} &= a_n^{(2)} T_2^n x^{(1)} + (1 - a_n^{(2)}) S_2 x, \\
x^{(3)} &= a_n^{(3)} T_3^n x^{(2)} + (1 - a_n^{(3)}) S_3 x, \\
&\vdots \\
x^{(N-1)} &= a_n^{(N-1)} T_{N-1}^n x^{(N-2)} + (1 - a_n^{(N-1)}) S_{N-1} x, \\
U_n x &= a_n^{(N)} T_N^n x^{(N-1)} + (1 - a_n^{(N)}) S_N x.
\end{aligned}$$

Then

$$\|U_n x - U_n y\| \le k_n^N \|x - y\|.$$

Set

$$\begin{aligned}
W_{n,m} &= U_{n+m-1} \circ U_{n+m-2} \circ \cdots \circ U_n, \quad m \ge 1, \\
b_{n,m} &= \|W_{n,m}(tx_n + (1-t)u) - (tW_{n,m}x_n + (1-t)W_{n,m}u)\|.
\end{aligned}$$

Then observing that $W_{n,m}x_n = x_{n+m}$, we get

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)u - v\| \\
&\le b_{n,m} + \|W_{n,m}(tx_n + (1-t)u) - v\| \\
&\le b_{n,m} + \left(\prod_{j=n}^{n+m-1} k_j^N \right) a_n(t) \\
&\le b_{n,m} + L_n a_n(t),
\end{aligned}$$

where $L_n = \prod_{j=n}^{\infty} k_j^N$. By Lemma 5 we have

$$\begin{aligned}
b_{n,m} &\le L_n \phi^{-1}(\|x_n - u\| - L_n^{-1}\|W_{n,m}x_n - u\|) \\
&\le L_n \phi^{-1}(\|x_n - u\| - \|x_{n+m} - u\| + (1 - L_n^{-1})d),
\end{aligned}$$

where $\phi\colon [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function depending only on the diameter of K and $\phi(0) = 0$. Since $\lim_{n\to\infty} L_n = 1$, it follows from Lemma 6 that $\lim_{n,m\to\infty} b_{n,m} = 0$. Therefore,

$$\limsup_{m\to\infty} a_m(t) \le \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} L_n a_n(t) = \liminf_{n\to\infty} a_n(t).$$

This completes the proof. □

Recall that a Banach space E has the *Kadec–Klee property* if for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Theorem 2 *Let K be a nonempty closed convex subset of a uniformly convex Banach space E such that its dual E^* has the Kadec–Klee property. Let $S_1, S_2, \ldots, S_N\colon K \to K$ be nonexpansive mappings, $T_1, T_2, \ldots, T_N\colon K \to K$ be asymptotically nonexpansive mappings with the sequence $\{k_n\}$ and suppose that*

$$F = \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \varnothing.$$

If (3), (8) and (18) hold, then the sequence $\{x_n\}$ defined by (1) converges weakly to a common fixed point of $S_1, S_2, \ldots, S_N$, $T_1, T_2, \ldots, T_N$.

Proof Let $q \in F$. Then by Lemma 6, $\lim_{n\to\infty} \|x_n - q\|$ exists. Since E is reflexive and $\{x_n\}$ is bounded sequence in K, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. Moreover $\lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$ and $\lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for all $i = 1, 2, \ldots, N$, by Lemma 9. From Lemma 1, we have that $(I - S_i)p = (I - T_i)p = 0$ for all $i = 1, 2, \ldots, N$. Thus, $p \in F$.

Now, we show that $\{x_n\}$ converges weakly to p. Suppose that $\{x_{n_k}\}$ is another subsequence of $\{x_n\}$ which converges weakly to some $p' \in K$. By the same method as above, we have $p' \in F$ and so $p, p' \in \omega_w(x_n)$. Then by Lemma 10,

$$\lim_{n\to\infty} \|tx_n + (1-t)p - p'\|$$

exists for all $t \in [0,1]$. Now, Lemma 4 guarantees that $p = p'$. As a result, the whole sequence $\{x_n\}$ converges weakly to p. This completes the proof. □

6 Some analogues and corollaries

With a little effort, we have the following analogues to Theorems 1 and 2.

Theorem 3 *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S_1, S_2, \ldots, S_N\colon K \to K$ be nonexpansive mappings,*

$T_1, T_2, \dots, T_N \colon K \to K$ be asymptotically nonexpansive mappings with the sequence $\{k_n\}$ and suppose that $\bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \varnothing$. Let $\{x_n\}$ be the sequence defined by

$$\left.\begin{aligned} x_1 &\in K, \\ x_n^{(0)} &= x_n, \\ x_n^{(1)} &= a_n^{(1)} T_1^n x_n^{(0)} + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)}, \\ x_n^{(2)} &= a_n^{(2)} T_2^n x_n^{(1)} + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)}, \\ &\vdots \\ x_n^{(N-1)} &= a_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + b_n^{(N-1)} S_{N-1} x_n + c_n^{(N-1)} u_n^{(N-1)}, \\ x_n^{(N)} &= a_n^{(N)} T_N^n x_n^{(N-1)} + b_n^{(N)} S_N x_n + c_n^{(N)} u_n^{(N)}, \\ x_{n+1} &= x_n^{(N)}, \quad n \geq 1, \end{aligned}\right\} \tag{24}$$

where $\{u_n^{(i)}\}$ are bounded sequences in K and $\{a_n^{(i)}\}_{n=1}^\infty$, $\{b_n^{(i)}\}_{n=1}^\infty$, $\{c_n^{(i)}\}_{n=1}^\infty \subset [0,1]$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for all $i = 1, 2, \dots, N$.

Suppose that $\sum_{n=1}^\infty c_n^{(i)} < \infty$ for all $i = 1, 2, \dots, N$,

(i) $\sum_{n=1}^\infty (k_n - 1) < \infty$,

(ii) there is $\delta > 0$ such that $\delta \leq a_n^{(i)} \leq 1 - \delta$ for all $n \geq 1, i = 1, 2, \dots, N$,

(iii) $\|x - T_i y\| \leq \|S_i x - T_i y\|$ for all $x, y \in K$ and $i = 1, 2, \dots, N$.

(a) If the family $\{S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N\}$ satisfies condition (B), then $\{x_n\}$ converges strongly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$.

(b) If the dual E^* has the Kadec–Klee property, then $\{x_n\}$ converges weakly to a common fixed point of $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N$.

Remark

1. If, moreover $S_1 = S_2 = \dots = S_N = S$, then by Lemma 7

$$\lim_{n\to\infty} \|S x_n - T_i x_n^{(i-1)}\| = 0 \quad \text{for all } i = 1, 2, \dots, N.$$

   The assumption (iii) in Theorem 3 can be weakened by assuming that there is $i_0 \in \{1, 2, \dots, N\}$ such that

$$\|x - T_{i_0} y\| \leq \|S x - T_{i_0} y\| \quad \text{for all } x, y \in K.$$

2. If, moreover $S_1 = S_2 = \dots = S_N = I$, then Theorems 2.3 and 2.9 of [18] become a corollary of Theorem 3.

3. Theorem 3 is not only an extension of [9] and [10] but also obtained under the different assumptions.

Theorem 4 *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N \colon K \to K$ be nonexpansive* *mappings and suppose that* $\bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \varnothing$*. Let* $\{x_n\}$ *be the sequence defined by*

$$\left.\begin{aligned} x_1 &\in K, \\ x_n^{(0)} &= x_n, \\ x_n^{(1)} &= a_n^{(1)} T_1 x_n^{(0)} + b_n^{(1)} S_1 x_n + c_n^{(1)} u_n^{(1)}, \\ x_n^{(2)} &= a_n^{(2)} T_2 x_n^{(1)} + b_n^{(2)} S_2 x_n + c_n^{(2)} u_n^{(2)}, \\ &\vdots \\ x_n^{(N-1)} &= a_n^{(N-1)} T_{N-1} x_n^{(N-2)} + b_n^{(N-1)} S_{N-1} x_n + c_n^{(N-1)} u_n^{(N-1)}, \\ x_n^{(N)} &= a_n^{(N)} T_N x_n^{(N-1)} + b_n^{(N)} S_N x_n + c_n^{(N)} u_n^{(N)}, \\ x_{n+1} &= x_n^{(N)}, \quad n \geq 1, \end{aligned}\right\} \tag{25}$$

where $\{u_n^{(i)}\}$ *are bounded sequences in* K *and* $\{a_n^{(i)}\}_{n=1}^{\infty}$*,* $\{b_n^{(i)}\}_{n=1}^{\infty}$*,* $\{c_n^{(i)}\}_{n=1}^{\infty} \subset [0,1]$ *such that* $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ *for all* $i = 1, 2, \ldots, N$*.*

Suppose that $\sum_{n=1}^{\infty} c_n^{(i)} < \infty$ *for all* $i = 1, 2, \ldots, N$*,*

(i) there is $\delta > 0$ *such that* $\delta \leq a_n^{(i)} \leq 1 - \delta$ *for all* $n \geq 1, i = 1, 2, \ldots, N$*,*

(ii) $\|x - T_i y\| \leq \|S_i x - T_i y\|$ *for all* $x, y \in K$ *and* $i = 1, 2, \ldots, N$*.*

(a) If the family $\{S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N\}$ *satisfies condition (B), then* $\{x_n\}$ *converges strongly to a common fixed point of* $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N$*.*

(b) If the dual E^* *has the Kadec–Klee property, then* $\{x_n\}$ *converges weakly to a common fixed point of* $S_1, S_2, \ldots, S_N, T_1, T_2, \ldots, T_N$*.*

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