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A Result on Segmenting Jungck–Mann Iterates

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Abstract

In this paper, following the concepts in [5, 7], we shall establish a convergence result in a uniformly convex Banach space using the Jungck–Mann iteration process introduced by Singh et al [13] and a certain general contractive condition. The authors of [13] established various stability results for a pair of nonself-mappings for both Jungck and Jungck–Mann iteration processes. Our result is a generalization and extension of that of [7] and its corollaries. It is also an improvement on the result of [7].

Key words: Jungck–Mann iteration process; uniformly convex Banach space.

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1 Introduction

Suppose that $A = (a_{nk})$ is an infinite, lower triangular, regular row-stochastic matrix, E a closed convex subset of a Banach space and T a continuous mapping of E into itself and $x_1 \in E$. Then, the general Mann iteration process $M(x_1, A, T)$ which was introduced in Mann [9] is defined by

$$v_n = \sum_{k=1}^{n} a_{nk} x_k, \quad x_{n+1} = T v_n, \quad n = 1, 2, \dots, \tag{1}$$

If A is the identity matrix, then each sequence of $M(x_1, A, T)$ becomes the sequence of Picard iterates of T at x_1. It was established in [9] that if either of the sequences $\{x_n\}$ a nd $\{v_n\}$ converges, then the other also converges to the same point, and their common limit is a fixed point of T.

In [5, 7], it is said that the matrix A is *segmenting* for the Mann process if $a_{n+1,k} = (1 - a_{n+1,n+1})a_{nk}$ for $k \leq n$. In this case, v_{n+1} lies on the segment joining v_n and Tv_n:

$$v_{n+1} = (1 - d_n)v_n + d_n Tv_n, \quad n = 1, 2, \ldots, \tag{2}$$

where $d_n = a_{n+1,n+1}$. A segmenting matrix is determined by its sequence of diagonal elements. Some authors including [3, 11, 12] have investigated the case $d_n = \lambda$, $0 < \lambda < 1$, while Mann [9] approximated the fixed points of continuous functions on a closed interval of the real line using the segmenting matrix determined by $d_n = \frac{1}{n}\ \forall\ n$. Dotson [6] considered the case when d_n is bounded away from 0 and 1. Groetsch [7] generalized the results of [3, 6, 9, 11, 12] in a uniformly convex Banach space by employing (2) and assuming that A is a segmenting matrix for which $\sum_{n=1}^{\infty} d_n(1 - d_n) = \infty$.

We shall give another definition of a segmenting matrix in the next section with a view to generalizing and extending Groetsch [7] and others mentioned earlier in this paper.

2 Preliminaries

Singh et al [13] introduced the following iteration process: Let $(E, \|.\|)$ be a normed linear space, $S, T \colon Y \to E$ and $T(Y) \subseteq S(Y)$. Then, for $x_0 \in Y$, consider the iteration process

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ satisfies

(i) $\alpha_0 = 1$,

(ii) $0 \leq \alpha_n \leq 1$ for $n > 0$,

(iii) $\sum \alpha_n = \infty$, and

(iv) $\sum_{j=0}^{n} \alpha_j \Pi_{i=j+1}^{n}(1 - \alpha_i + a\alpha_i)$ converges.

The iteration process (3) is called the *Jungck–Mann iteration.*

For $Y = E$, $S = I$ (identity operator) in (3) with $\{\alpha_n\}_{n=0}^{\infty}$ satisfying (i)–(iv), then we have the Mann iteration process introduced by Mann [9]. Also, if in (3), $Y = E$, $S = I$ (identity operator) and $\alpha_n = 1$, then we obtain the Jungck iteration introduced by Jungck [8].

Following (3), we shall generalize and extend Groetsch [7] and others mentioned earlier in this paper by assuming that A is a segmenting matrix for which

$$Sv_{n+1} = (1 - d_n)Sv_n + d_n Tv_n, \quad n = 1, 2, \ldots, \tag{$\star$}$$

such that $\sum_{n=1}^{\infty} d_n(1-d_n) = \infty$ and $S, T\colon C \to C$ are selfmappings on a nonempty convex subset C of a uniformly convex Banach space E. The operators S and T are assumed to have a common fixed point and satisfy in addition the contractive condition

$$\|Tx - Ty\| \leq \|Sx - Sy\|, \quad \forall x, y \in C. \tag{$\star\star$}$$

If $S = I$ (identity operator) in ($\star$), then we obtain (2) and if $S = I$ in ($\star\star$) then we have $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$ (that is, T becomes a nonexpansive mapping).

We shall establish our main result in the next section. However, the following lemma is required in the sequel.

Lemma 2.1 (Groetsch [7]) *Let X be a uniformly convex Banach space and let $x, y \in X$. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon > 0$, then*

$$\|\lambda x + (1-\lambda)y\| \leq 1 - 2\lambda(1-\lambda)\delta(\epsilon)$$

for $0 \leq \lambda < 1$ and $\delta(\epsilon) > 0$.

The proof of this Lemma is contained in [4, 7].

3 The Main Result

Theorem 3.1 *Let C be a convex subset of a uniformly convex Banach space E and $S, T\colon C \to C$ selfmappings satisfying condition ($\star\star$) and $T(C) \subseteq S(C)$. Suppose that S and T have at least a common fixed point. Let $\{Sv_n\}_{n=1}^{\infty}$ be the sequence defined by ($\star$). Then, the sequence $\{(S-T)v_n\}_{n=1}^{\infty}$ converges strongly to 0 for each $x_1 \in C$ such that $\sum_{n=1}^{\infty} d_n(1-d_n) = \infty$.*

Proof If p is a common fixed point of S and T (i.e. $Sp = Tp = p$), then

$$\begin{aligned}
\|Sv_{n+1} - p\| &= \|(1-d_n)Sv_n + d_n Tv_n - (1 - d_n + d_n)p\| \\
&= \|(1-d_n)(Sv_n - p) + d_n(Tv_n - p)\| \\
&\leq (1-d_n)\|Sv_n - p\| + d_n\|Tv_n - p\| \\
&= (1-d_n)\|Sv_n - p\| + d_n\|Tv_n - Tp\| \\
&\leq (1-d_n)\|Sv_n - p\| + d_n\|Sv_n - Sp\| \\
&= (1-d_n)\|Sv_n - p\| + d_n\|Sv_n - p\| \\
&= \|Sv_n - p\| \leq \|Sv_{n-1} - p\| \leq \cdots \leq \|Sv_1 - p\|,
\end{aligned} \tag{4}$$

from which we have that the sequence $\{Sv_n - p\}_{n=1}^{\infty}$ is decreasing.

Now,

$$\begin{aligned}
&\|(S-T)v_n\| = \|Sv_n - Tv_n\| \leq \|Sv_n - p\| + \|p - Tv_n\| \\
&\quad = \|Sv_n - p\| + \|Tp - Tv_n\| \leq \|Sv_n - p\| + \|Sp - Sv_n\| = 2\|Sv_n - p\|.
\end{aligned}$$

Suppose on the contrary that $\{(S-T)v_n\}_{n=1}^{\infty}$ does not converge to 0. Since $\|Sv_n - Tv_n\| \leq 2\|Sv_n - p\|$, we may assume that there is an $a > 0$, $a \in (0,1)$ such that $\|Sv_n - p\| \geq a$ for any n. If $\{(S-T)v_n\}_{n=1}^{\infty}$ does not converge to 0, then there is an $\epsilon > 0$ such that $\|Sv_n - Tv_n\| \geq \epsilon$ for any n.

Let

$$b = 2\delta\left(\frac{\epsilon}{\|Sv_1 - p\|}\right), \quad x_n = \frac{Sv_n - p}{\|Sv_n - p\|} \quad \text{and} \quad y_n = \frac{Tv_n - p}{\|Sv_n - p\|}.$$

Then, we have

$$\|x_n\| = \left\|\left(\frac{Sv_n - p}{\|Sv_n - p\|}\right)\right\| \leq \frac{\|Sv_n - p\|}{\|Sv_n - p\|} = 1$$

and

$$\|y_n\| = \left\|\left(\frac{Tv_n - p)}{\|Sv_n - p\|}\right)\right\| \leq \frac{\|Tv_n - Tp)\|}{\|Sv_n - p\|} \leq \frac{\|Sv_n - Sp\|}{\|Sv_n - p\|} = \frac{\|Sv_n - p\|}{\|Sv_n - p\|} = 1.$$

Hence, we have by $(\star)$ that

$$\begin{aligned}
\|Sv_{n+1} - p\| &= \|(1-d_n)Sv_n + d_nTv_n - (1 - d_n + d_n)p\| \\
&= \|(1-d_n)(Sv_n - p) + d_n(Tv_n - p)\| \\
&= \left\|(\|Sv_n - p\|)\left[(1-d_n)\frac{(Sv_n - p)}{\|Sv_n - p\|} + d_n\frac{(Tv_n - p)}{\|Sv_n - p\|}\right]\right\| \\
&= \|(\|Sv_n - p\|)[(1-d_n)x_n + d_ny_n]\| \\
&\leq \|Sv_n - p\| \; \|(1-d_n)x_n + d_ny_n\|. \qquad (5)
\end{aligned}$$

Using (4) and Lemma 2.1 in (5) yield

$$\begin{aligned}
&\|Sv_{n+1} - p\| \leq \\
&\leq [1 - d_n(1-d_n)b]\|Sv_n - p\| \\
&= \|Sv_n - p\| - bd_n(1-d_n)\|Sv_n - p\| \\
&\leq \|Sv_{n-1} - p\| - bd_{n-1}(1-d_{n-1})\|Sv_{n-1} - p\| - bd_n(1-d_n)\|Sv_n - p\| \\
&\leq \|Sv_{n-1} - p\| - bd_{n-1}(1-d_{n-1})\|Sv_n - p\| - bd_n(1-d_n)\|Sv_n - p\| \\
&= \|Sv_{n-1} - p\| - b[d_{n-1}(1-d_{n-1}) + d_n(1-d_n)]\|Sv_n - p\|.
\end{aligned}$$

Repeating this process inductively leads to

$$a \leq \|Sv_{n+1} - p\| \leq \|Sv_1 - p\|$$
$$- b\Big[d_1(1-d_1)\|Sv_n - p\| + d_2(1-d_2)\|Sv_n - p\| + \cdots + d_n(1-d_n)\|Sv_n - p\|\Big]$$
$$= \|Sv_1 - p\| - b\sum_{j=1}^{n} d_j(1-d_j)\|Sv_n - p\| \leq \|Sv_1 - p\| - ab\sum_{j=1}^{n} d_j(1-d_j).$$

Therefore, we obtain

$$a\left[1+b\sum_{j=1}^{n} d_j(1-d_j)\right] \le \|Sv_1 - p\|,$$

from which it follows that

$$a \le \frac{\|Sv_1 - p\|}{1+b\sum_{j=1}^{n} d_j(1-d_j)} \to 0 \quad \text{as } n \to \infty,$$

leading to a contradiction. Therefore, we have $a = 0$. Hence,

$$\lim_{n\to\infty} \|Sv_n - Tv_n\| = 0.$$

Remark 3.1 Theorem 3.1 is also a generalization of the results of [3, 6, 7, 9, 11, 12].

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