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Classes of Filters in Generalizations of Commutative Fuzzy Structures*

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Abstract

Bounded commutative residuated lattice ordered monoids ($R\ell$-monoids) are a common generalization of BL-algebras and Heyting algebras, i.e. algebras of basic fuzzy logic and intuitionistic logic, respectively. In the paper we develop the theory of filters of bounded commutative $R\ell$-monoids.

Key words: Residuated ℓ-monoid, deductive system, BL-algebra, MV-algebra, Heyting algebra, filter.

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1 Introduction

BL-algebras have been introduced by P. Hájek as an algebraic counterpart of the basic fuzzy logic BL [5]. Omitting the requirement of pre-linearity in the definition of a BL-algebra, one obtains the definition of a bounded commutative residuated lattice ordered monoid ($R\ell$-monoid). Nevertheless, bounded commutative $R\ell$-monoids are a generalization not only of BL-algebras but also of Heyting algebras which are an algebraic counterpart of the intuitionistic propositional logic. Therefore, bounded commutative $R\ell$-monoids could be taken as an algebraic semantics of a more general logic than Hájek's fuzzy logic. It is

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known that every BL-algebra (and consequently every MV-algebra [2], or equivalently, every Wajsberg algebra [4]) is a subdirect product of linearly ordered BL-algebras. Moreover, a bounded commutative $R\ell$-monoid is a subdirect product of linearly ordered $R\ell$-monoids if and only if it is a BL-algebra [13]. On the other side, bounded commutative $R\ell$-monoids which need not be BL-algebras can be constructed from BL-algebras by means of other natural operations, e.g. by means of pasting, i.e. ordinal sums. For example, the pasting of Wajsberg algebras which are not linearly ordered gives bounded commutative $R\ell$-monoids which are not BL-algebras [8, 9].

In both BL-algebras and bounded commutative $R\ell$-monoids, filters coincide with deductive systems of those algebras and are exactly the kernels of their congruences. Various types of filters of BL-algebras were studied in [19], [7] and [11]. Boolean filters of bounded commutative $R\ell$-monoids were investigated in [14].

In this paper we further develop the theory of filters of bounded commutative $R\ell$-monoids and among others, we generalize some results of [7] and [11].

For concepts and results concerning MV-algebras, BL-algebras and Heyting algebras see for instance [2], [5], [1].

2 Preliminaries

A *bounded commutative $R\ell$-monoid* is an algebra $M = (M;\ \odot, \vee, \wedge, \to, 0, 1)$ of type $\langle 2,2,2,2,0,0\rangle$ satisfying the following conditions:

($R\ell1$) $(M; \odot, 1)$ is a commutative monoid.

($R\ell2$) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.

($R\ell3$) $x \odot y \le z$ if and only if $x \le y \to z$, for any $x, y, z \in M$.

($R\ell4$) $x \odot (x \to y) = x \wedge y$, for any $x, y \in M$.

In the sequel, by an *$R\ell$-monoid* we will mean a *bounded commutative $R\ell$-monoid.*

On any $R\ell$-monoid M let us define a unary operation negation $^-$ by $x^- := x \to 0$ for any $x \in M$.

Bounded commutative $R\ell$-monoids are special cases of residuated lattices, more precisely (see for instance [3]), they are exactly commutative integral generalized BL-algebras in the sense of [10].

The above mentioned algebras can be characterized in the class of all $R\ell$-monoids as follows: An $R\ell$-monoid M is

a) a BL-algebra if and only if M satisfies the identity of pre-linearity $(x \to y) \vee (y \to x) = 1$;

b) an MV-algebra if and only if M fulfills the double negation law $x^{--} = x$;

c) a Heyting algebra if and only if the operation "$\odot$" is idempotent.

Lemma 2.1 *See [15] and [16]. In any bounded commutative $R\ell$-monoid M we have for any $x, y, z \in M$:*

(1) $1 \to x = x$.

(2) $x \leq y \Longleftrightarrow x \to y = 1$.

(3) $x \odot y \leq x \wedge y$.

(4) $x \leq y \to x$.

(5) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$.

(6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

(7) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

(8) $x \leq x^{--},\ x^{-} = x^{---}$.

(9) $x \leq y \Longrightarrow y^{-} \leq x^{-}$.

(10) $(x \odot y)^{-} = y \to x^{-} = y^{--} \to x^{-} = x \to y^{-} = x^{--} \to y^{-}$.

(11) $x \leq y \Longrightarrow z \to x \leq z \to y,\ y \to z \leq x \to z$.

(12) $x \to y \leq y^{-} \to x^{-}$.

(13) $x \vee y \leq ((x \to y) \to y) \wedge ((y \to x) \to x)$.

(14) $x \to y \leq (y \to z) \to (x \to z)$.

(15) $x \to y \leq (z \to x) \to (z \to y)$.

A non-empty subset F of an $R\ell$-monoid M is called a *filter* of M if

(F1) $x, y \in F$ imply $x \odot y \in F$;

(F2) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

A subset D of an $R\ell$-monoid M is called a *deductive system* of M if

(i) $1 \in D$;

(ii) $x \in D,\ x \to y \in D$ imply $y \in D$.

Proposition 2.2 *[3]. Let H be a non-empty subset of M. Then H is a filter of M if and only if H is a deductive system of M.*

By [18], filters of commutative $R\ell$-monoids are exactly the kernels of their congruences. If F is a filter of M, then F is the kernel of the unique congruence $\Theta(F)$ such that $\langle x, y\rangle \in \Theta(F)$ if and only if $(x \to y) \wedge (y \to x) \in F$, for any $x, y \in M$. Hence we will consider quotient $R\ell$-monoids M/F of $R\ell$-monoids M by their filters F.

A filter F of M is called *maximal* if F is a proper filter of M and is not a proper subset of any proper filter of M.

3 Implicative filters

Let M be an $R\ell$-monoid and F a subset of M. Then F is called an *implicative filter* of M if

(1) $1 \in F$;

(2) $x \to (y \to z) \in F$, $x \to y \in F$ imply $x \to z \in F$.

Proposition 3.1 *Every implicative filter of an $R\ell$-monoid M is a filter of M.*

Proof Let $\emptyset \neq F \subseteq M$ satisfy conditions (1) and (2) and let $x, y \in M$ be such that x, $x \to y \in F$. Then $1 \to (x \to y) \in F$, $1 \to x \in F$, hence $y = 1 \to y \in F$. □

If F is a filter of an $R\ell$-monoid M and $a \in M$, put

$$M_a := \{x \in M : a \to x \in F\}.$$

Theorem 3.2 *Let M be an $R\ell$-monoid and F be a filter of M. Then F is an implicative filter of M if and only if M_a is a filter of M for every $a \in M$.*

Proof Let F be an implicative filter of M and $a \in M$. Then $1 = a \to 1 \in M$, thus $1 \in M_a$. Further, suppose that x, $x \to y \in M_a$, i.e. $a \to x \in F$ and $a \to (x \to y) \in F$. Then we get $a \to y \in F$, and hence $y \in M_a$. That means, M_a is a filter of M for arbitrary $a \in M$.

Conversely, let M_a be a filter of M for each $a \in M$. Suppose that $x \to (y \to z) \in F$ and $x \to y \in F$. Then $y \to z \in M_x$ and $y \in M_x$, hence $z \in M_x$ and therefore $x \to z \in F$. That means, F is implicative. □

Theorem 3.3 *Let F be a filter of an $R\ell$-monoid M. Then the following conditions are equivalent:*

(a) F is an implicative filter of M.

(b) $y \to (y \to x) \in F$ implies $y \to x \in F$, for any $x, y \in M$.

(c) $z \to (y \to x) \in F$ implies $(z \to y) \to (z \to x) \in F$, for any $x, y, z \in M$.

(d) $z \to (y \to (y \to x)) \in F$ and $z \in F$ imply $y \to x \in F$, for any $x, y, z \in M$.

(e) $x \to (x \odot x) \in F$, for any $x \in M$.

Proof (a) ⇒ (b): Suppose that F is an implicative filter of M, $x, y \in M$ and $y \to (y \to x) \in F$. Then since $y \to y = 1 \in F$, we obtain $y \to x \in F$.

(b) ⇒ (c): Let F be a filter of M satisfying the condition (b), $x, y, z \in M$ and $z \to (y \to x) \in F$. Then $z \to (z \to ((z \to y) \to x)) = z \to ((z \to y) \to (z \to x)) \geq z \to (y \to x) \in F$, thus $z \to (z \to ((z \to y) \to x)) \in F$. From this we have $z \to ((z \to y) \to x) \in F$, that means $(z \to y) \to (z \to x) \in F$.

(c) ⇒ (d): Suppose that a filter F satisfies the condition (c). Let $z \to (y \to (y \to x)) \in F$ and $z \in F$. Then also $y \to (y \to x) \in F$. At the same time, $y \to x = (y \to y) \to (y \to x)$, thus $y \to x \in F$.

(d) ⇒ (a): Let a filter F fulfill the condition (d). Let $x \to (y \to z) \in F$ and $x \to y \in F$. Then $x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z))$, hence $(x \to y) \to (x \to (x \to z)) \in F$, and therefore $x \to z \in F$.

(a) ⇒ (e): Let F be an implicative filter of M. Then $x \to (x \to (x \odot x)) = (x \odot x) \to (x \odot x) = 1 \in F$. Further, $x \to x = 1 \in F$, and hence we obtain $x \to (x \odot x) \in F$.

(e) ⇒ (a): Let a filter F satisfy the condition (e) and let $x \to (y \to z) \in F$ and $x \to y \in F$. Then $(x \to (y \to z)) \odot (x \to y) \odot x \odot x \leq (y \to z) \odot y \leq z$, hence $(x \to (y \to z)) \odot (x \to y) \leq (x \odot x) \to z$, and thus $(x \odot x) \to z \in F$. Further, $x \to (x \odot x) \in F$, $(x \odot x) \to x = 1 \in F$, therefore from $(x \odot x) \to z \in F$, we obtain $x \to z \in F$. □

Using the proof (a) ⇒ (e) in the preceding theorem, we have as an immediate consequence:

Theorem 3.4 *If F is a filter of an $R\ell$-monoid M, then F is an implicative filter if and only if the quotient $R\ell$-monoid M/F is a Heyting algebra.*

Proposition 3.5 *If F_1 and F_2 are filters of an $R\ell$-monoid M, $F_1 \subseteq F_2$ and F_1 is an implicative filter of M, then F_2 is also an implicative filter of M.*

Proof Suppose that F_1 and F_2 are filters of an $R\ell$-monoid M, $F_1 \subseteq F_2$ and F_1 is implicative. Then, by Theorem 3.3, $x \to x \odot x \in F_1 \subseteq F_2$ for any $x \in M$, and therefore F_2 is also implicative. □

Let M be an $R\ell$-monoid and F a subset of M. Then F is called a *positive implicative filter* of M if

(1) $1 \in F$;

(3) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$, for any $x, y, z \in M$.

Proposition 3.6 *Every positive implicative filter of an $R\ell$-monoid M is a filter of M.*

Proof Let $x \in F$ and $x \to y \in F$. Then $x \to ((y \to 1) \to y) = x \to (1 \to y) = x \to y$, hence $x \to ((y \to 1) \to y) \in F$, and thus $y \in F$. □

Proposition 3.7 *Every positive implicative filter of M is an implicative filter of M.*

Proof Let F be a positive implicative filter of M, $x, y, z \in M$, $x \to (y \to z) \in F$ and $x \to y \in F$. We have $(x \to y) \to (x \to (x \to z)) \geq y \to (x \to z) = x \to (y \to z)$, hence $(x \to y) \to (x \to (x \to z)) \in F$, and thus also $x \to (x \to z) \in F$.

Since $((x \to z) \to z) \to (x \to z) \geq x \to (x \to z)$, then we get $((x \to z) \to z) \to (x \to z) \in F$. Further, $1 \to (((x \to z) \to z) \to (x \to z)) = ((x \to z) \to z) \to (x \to z)$, and since $1 \to (((x \to z) \to z) \to (x \to z)) \in F$ and $1 \in F$, we obtain $x \to z \in F$.

Therefore F is an implicative filter. □

Theorem 3.8 *Let F be a filter of an $R\ell$-monoid M. Then the following conditions are equivalent:*

(a) F is a positive implicative filter of M.

(b) $(x \to y) \to x \in F$ implies $x \in F$, for any $x, y \in M$.

(c) $(x^- \to x) \to x \in F$, for any $x \in M$.

Proof (a) $\Rightarrow$ (b): Let F be a positive implicative filter of M and $(x \to y) \to x \in F$. Then since $1 \to ((x \to y) \to x) = (x \to y) \to x \in F$ and $1 \in F$, we get $x \in F$.

(b) $\Rightarrow$ (a): Let a filter F satisfy the condition (b) and let $x \to ((y \to z) \to y) \in F$ and $x \in F$. Then $(y \to z) \to y \in F$, and therefore $y \in F$. Hence F is a positive implicative filter of M.

(b) $\Rightarrow$ (c): Let F be a filter of M and $x \in M$. Then $(((x^- \to x) \to x) \to 0) \to ((x^- \to x) \to x) = (x^- \to x) \to ((((x^- \to x) \to x) \to 0) \to x) \geq (((x^- \to x) \to x) \to 0) \to x^- = ((x^- \to x) \to x) \to 0) \to (x \to 0) \geq x \to ((x^- \to x) \to x) = 1 \in F$, thus $(((x^- \to x) \to x) \to 0) \to ((x^- \to x) \to x) \in F$, and hence $(x^- \to x) \to x \in F$.

(c) $\Rightarrow$ (b): Let a filter F satisfy condition (c). Let $(x \to y) \to x \in F$. We have $(x \to y) \to x \leq (x \to 0) \to x = x^- \to x$, hence $x^- \to x \in F$. By the assumption, $(x^- \to x) \to x \in F$, thus $x \in F$. Therefore F satisfies the condition (b). □

Proposition 3.9 *If F_1 and F_2 are filters of an $R\ell$-monoid M, F_1 is a positive implicative filter and $F_1 \subseteq F_2$, then F_2 is also a positive implicative filter of M.*

Proof Let $F_1 \subseteq F_2$ and F_1 be positive implicative. Then for any $x \in M$ we get $(x^- \to x) \to x \in F_1$, thus $(x^- \to x) \to x \in F_2$. Therefore, by Theorem 3.8, F_2 is a positive implicative filter of M. □

Theorem 3.10 *Let M be an $R\ell$-monoid. Then the following conditions are equivalent:*

(a) M is a Heyting algebra.

(b) Every filter of M is implicative.

(c) $\{1\}$ is an implicative filter of M.

Proof (a) $\Rightarrow$ (c): It follows from Theorem 3.4.

(a) $\Rightarrow$ (b): Let M be an idempotent $R\ell$-monoid, F be a filter of M, and $x \in M$. Then $x \to (x \odot x) = x \to x = 1 \in F$, hence by Theorem 3.3, F is an implicative filter.

(b) $\Rightarrow$ (c): It is obvious. □

Proposition 3.11 *Let F be an implicative filter of an $R\ell$-monoid M. Then the following conditions are equivalent:*

(a) F is a positive implicative filter of M.

(b) $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$, for any $x, y \in M$.

Proof (a) ⇒ (b): Let F be a positive implicative filter of M and $(x \to y) \to y \in F$. Since $x \leq (y \to x) \to x$, we get $((y \to x) \to x) \to y \leq x \to y$. Hence $(x \to y) \to y \leq (y \to x) \to ((x \to y) \to x) = (x \to y) \to ((y \to x) \to x) \leq (((y \to x) \to x) \to y) \to ((y \to x) \to x)$, and thus $(((y \to x) \to x) \to y) \to ((y \to x) \to x) \in F$. Consequently, also $1 \to ((((y \to x) \to x) \to y) \to ((y \to x) \to x)) \in F$, and since F is a positive implicative filter, we get $(y \to x) \to x \in F$.

(b) ⇒ (a): Let an implicative filter F satisfy the condition (b) and let $x \in F$ and $x \to ((y \to z) \to y) \in F$. Then also $(y \to z) \to y \in F$. Further, $(y \to z) \to y \leq (y \to z) \to ((y \to z) \to z)$, hence $(y \to z) \to ((y \to z) \to z) \in F$. Since F is implicative, $(y \to z) \to z \in F$. Then, by the assumption, also $(z \to y) \to y \in F$. Further, $z \leq y \to z$, hence $(y \to z) \to y \leq z \to y$, thus $z \to y \in F$. We have shown $(z \to y) \to y \in F$, therefore $y \in F$. □

Theorem 3.12 *Let M be an $R\ell$-monoid. Then the following conditions are equivalent:*

(a) $\{1\}$ is a positive implicative filter.

(b) Every filter of M is positive implicative.

(c) $M(a) := \{x \in M : a \leq x\}$ is a positive implicative filter of M, for every $a \in M$.

(d) $(x \to y) \to x = x$, for any $x, y \in M$.

(e) M is a Boolean algebra.

Proof (a) ⇒ (b): It follows from Proposition 3.9.

(b) ⇒ (c): Let $a \in M$. Then $1 \in M(a)$. Assume that $x, x \to y \in M(a)$, i.e. $a \to x = 1$, $a \to (x \to y) = 1$. Since by the assumption, $\{1\}$ is a positive implicative filter of M, we obtain $a \to y = 1$, hence $y \in M(a)$. That means $M(a)$ is a filter of M which is also positive implicative.

(c) ⇒ (d): If $x, y \in M$, then $(x \to y) \to x \in M((x \to y) \to x)$, therefore $(x \to y) \to x \leq x$ by Theorem 3.8. Moreover, $x \leq (x \to y) \to x$, i.e. $(x \to y) \to x = x$.

(d) ⇒ (a): It follows from Theorem 3.8.

(d) ⇒ (e): Since $(x \to y) \to x = x$, we obtain $(y \to x) \to x = (y \to x) \to ((x \to y) \to x) \geq (x \to y) \to y$, and similarly, $(x \to y) \to y \geq (y \to x) \to x$. Hence $x^{--} = (x \to 0) \to 0 = (0 \to x) \to x = 1 \to x = x$ and therefore by [12], M is an MV-algebra. Then by [7, Lemma 3.16], furthermore M is a Boolean algebra.

(e) ⇒ (d): Since M is a Boolean algebra, x^- is the lattice complement of x in M, and so $x \vee x^- = 1$. This implies, by [7, Lemma 3.16], $(x \to y) \to x = x$ for any $x, y \in M$. □

Theorem 3.13 *If F is a filter of an $R\ell$-monoid M, then the following conditions are equivalent:*

(a) F is a maximal and positive implicative filter of M.

(b) F is a maximal and implicative filter of M.

(c) If $x, y \in M \setminus F$, then $x \to y \in F$ and $y \to x \in F$.

(d) M/F is a two-element Boolean algebra.

Proof (a) ⇒ (b): It is obvious.

(b) ⇒ (c): Let F be a maximal and implicative filter of M. By Theorem 3.2, $M_y = \{a \in M : y \to a \in F\}$ is a filter of M. If $b \in F$, then from $b \leq y \to b$ it follows that $y \to b \in F$, thus $b \in M_y$. Hence $F \subseteq M_y$. Since F is a maximal filter of M and $y \notin F$, we have $M_y = M$. Therefore $y \to x \in F$. The assumption $x \notin F$ analogously implies $x \to y \in F$.

(c) ⇒ (a): Let a filter F satisfy the condition (c). Suppose that F is not positive implicative. Then by Theorem 3.8, there are $x, y \in M$ such that $x \notin F$ and $(x \to y) \to x \in F$. If $y \in F$, then $x \to y \in F$, and hence $x \in F$, a contradiction. If $y \notin F$, then by (c), $x \to y \in F$, a contradiction. Hence F is a positive implicative filter of M. We will prove that F is also a maximal filter of M. If $a \notin F$, then by the preceding part of the proof, $F \cup \{a\} \subseteq M_a$. We will show that M_a is the least filter of M containing $F \cup \{a\}$. Let G be a filter of M such that $F \cup \{a\} \subseteq G$. If $x \in M_a$, then $a \to x \in F \subseteq G$, and since $a \in G$, we have $x \in G$. Therefore $M_a \subseteq G$. Consider any element $z \in M$. If $z \in F$, then $z \in M_a$. If $z \notin F$, then since also $a \notin F$, the assumption (c) gives $a \to z \in M_a$. Hence $M_a = M$, and therefore F is a maximal filter of M.

(c) ⇒ (d): It is obvious. □

A filter F of an $R\ell$-monoid M is called

a) *Boolean* if $x \vee x^- \in F$ for every $x \in M$;

b) *semi-Boolean* if $(x \wedge x^-)^- \in F$ for every $x \in M$.

Proposition 3.14 *[14, Theorem 3.2]. If F is a filter of an $R\ell$-monoid M, then F is Boolean if and only if M/F is a Boolean algebra.*

Proposition 3.15 *Every Boolean filter of M is semi-Boolean.*

Proof Let $x \in M$. Then $x^- \leq (x \wedge x^-)^-$ and $x \leq x^{--} \leq (x \wedge x^-)^-$, hence $x \vee x^- \leq (x \wedge x^-)^-$. □

Example 3.16 Let $M = \{0, a, b, c, 1\}$ be the lattice with the diagram in Fig. 1, and let $\odot = \wedge$ and $\to$ be defined in the corresponding table in Fig. 1.

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

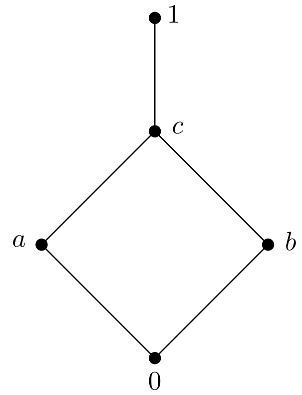

Fig. 1

Then $M = (M; \vee, \wedge, \odot, \to, 0, 1)$ is an $R\ell$-monoid (which is not a BL-algebra). The filter $F = \{1\}$ is semi-Boolean, but it is not Boolean.

Theorem 3.17 *a) Let M be an $R\ell$-monoid. Then every Boolean filter of M is positive implicative and every positive implicative filter of M is semi-Boolean.*

b) If an $R\ell$-monoid M satisfies condition

$$((x \to x^-) \to x^-) \wedge ((x^- \to x) \to x) = x \vee x^-, \text{ for any } x \in M, \qquad (*)$$

then Boolean and positive implicative filters of M coincide.

Proof a) Let M be an $R\ell$-monoid, let F be a Boolean filter of M and let $x \in M$. Then by Lemma 2.1, $x \vee x^- \le ((x \to x^-) \to x^-) \wedge ((x^- \to x) \to x)$, hence $((x \to x^-) \to x^-) \wedge ((x^- \to x) \to x) \in F$, and therefore $(x^- \to x) \to x \in F$. That means F is positive implicative.

Let now F be an arbitrary positive implicative filter of M and $x \in M$. Then $(x^{--} \to x^-) \to x^- \in F$ and by Lemma 2.1, $(x^{--} \to x^-) \to x^- = (x \to x^-) \to x^- = ((x \to x^-) \odot x)^- = (x \wedge x^-)^-$. Thus F is a semi-Boolean filter.

b) Let an $R\ell$-monoid M satisfy condition $(*)$ and let F be a positive implicative filter of M. Then a fortiori F is also implicative, hence $x \to (x \odot x) \in F$ for every $x \in M$. We have $(x \to x^-) \to x^- = (x \to (x \to 0)) \to (x \to 0) = ((x \odot x) \to 0) \to (x \to 0) \ge x \to (x \odot x)$, hence $(x \to x^-) \to x^- \in F$, and thus also $x \vee x^- = ((x \to x^-) \to x^-) \wedge ((x^- \to x) \to x) \in F$. Therefore F is a Boolean filter. $\square$

As an immediate consequence we get the following theorem.

Theorem 3.18 *[11, Theorem 2]. Boolean and positive implicative filters of any BL-algebra coincide.*

Proof If M is a BL-algebra, then by [5, Lemma 2.3.4(8)], $((x \to y) \to y) \wedge ((y \to x) \to x) = x \vee y$, for every $x, y \in M$. $\square$

Let F be a filter of an $R\ell$-monoid M. Then F is called an *implicative deductive system* if $x \to (z^- \to y) \in F$ and $y \to z \in F$ imply $x \to z \in F$, for any x, y, $z \in M$.

Theorem 3.19 *[14, Theorem 3.2]. Let F be a filter of an $R\ell$-monoid M. Then F is an implicative deductive system if and only if F is a Boolean filter.*

Remark 3.20 Now we can rephrase Theorem 3.17 in this way. Let M be an $R\ell$-monoid. Then every implicative deductive system of M is a positive implicative filter and every positive implicative filter of M is semi-Boolean. If M satisfies the condition $(*)$, then implicative deductive systems and positive implicative filters of M coincide.

Theorem 3.21 *If F is a maximal and (positive) implicative filter of an $R\ell$-monoid M, then F is Boolean.*

Proof Let F be a maximal and (positive) implicative filter of M. Then by Theorem 3.13, M/F is a two element $R\ell$-monoid, hence a two element Boolean algebra. Consequently, by Proposition 3.14, F is a Boolean filter. □

Theorem 3.22 *If F is a maximal filter of an $R\ell$-monoid M, then the following conditions are equivalent:*

(a) F is a Boolean filter.

(b) F is a positive implicative filter.

(c) F is an implicative filter.

(d) F is an implicative deductive system.

Proof It follows from Theorems 3.17 and 3.21 and from Remark 3.20. □

Let M be an $R\ell$-monoid. If F is a proper filter of M, denote

$$F^- := \{x \in M : x \leq y^- \text{ for some } y \in F\}.$$

By [14, Proposition 3.4], $F \cup F^-$ is a subalgebra of M for every proper filter F of M.

An $R\ell$-monoid M is called *bipartite* if $M = F \cup F^-$ for some maximal filter F of M.

By [14, Theorem 3.6], M is bipartite if and only if M contains a proper Boolean filter.

An $R\ell$-monoid M is said to be *strongly bipartite* if $M = F \cup F^-$ for every maximal filter F of M.

If M is an $R\ell$-monoid, denote by $B(M)$ the intersection of all Boolean filters of M. Obviously $B(M)$ is the least Boolean filter of M.

Further, denote by $\mathrm{Rad}(M)$ the *radical of* M, i.e. the intersection of all maximal filters of M.

Theorem 3.23 *[14, Theorem 3.8]. If M is an $R\ell$-monoid, then the following conditions are equivalent:*

(a) M is strongly bipartite.

(b) Every maximal filter of M is Boolean.

(c) $B(M) \subseteq \mathrm{Rad}(M)$.

The following theorem is an immediate consequence of Theorems 3.22 and 3.23.

Theorem 3.24 *If M is an $R\ell$-monoid, then the following conditions are equivalent:*

(a) *M is strongly bipartite.*

(b) $B(M) \subseteq \mathrm{Rad}(M)$.

(c) *Every maximal filter of M is Boolean.*

(d) *Every maximal filter of M is positive implicative.*

(e) *Every maximal filter of M is implicative.*

4 Fantastic filters

Let M be an $R\ell$-monoid and F a subset of M. Then F is called a *fantastic filter* of M if

(1) $1 \in F$;

(4) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$, for any $x, y, z \in M$.

Proposition 4.1 *Every fantastic filter of M is a filter of M.*

Proof Let F be a fantastic filter of M and $x, y \in M$. If x, $x \to y \in F$, then also $x \in F$ and $x \to (1 \to y) = x \to y \in F$, and thus by (4), $y \in F$. □

Theorem 4.2 *A filter F of an $R\ell$-monoid M is fantastic if and only if*

(5) *$y \to x \in F$ implies $((x \to y) \to y) \to x \in F$, for every $x, y \in M$.*

Proof Let F be a fantastic filter of M, $x, y \in M$ and $y \to x \in F$. Then $1 \to (y \to x) = y \to x \in F$ and $1 \in F$, hence $((x \to y) \to y) \to x \in F$.

Conversely, let a filter F satisfy the condition (5) and let $z \to (y \to x) \in F$ and $z \in F$. Then $y \to x \in F$, therefore also $((x \to y) \to y) \to x \in F$. □

Theorem 4.3 *Every positive implicative filter of an $R\ell$-monoid M is a fantastic filter of M.*

Proof Suppose F is a positive implicative filter of M and $x, y \in M$ are such that $y \to x \in F$. We have $x \leq ((x \to y) \to y) \to x$, thus

$$(((x \to y) \to y) \to x) \to y \leq x \to y.$$

Further, $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to (((x \to y) \to y) \to x) = ((x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$.

By the assumption $y \to x \in F$, hence also

$$((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \in F.$$

Since F is positive implicative, we get $((x \to y) \to y) \to x \in F$, and hence F is a fantastic filter. □

Theorem 4.4 *If F is a filter of an $R\ell$-monoid M, then the following conditions are equivalent:*

(a) F is a fantastic filter of M.

(b) $x^{--} \to x \in F$, for every $x \in M$.

(c) $x \to u \in F$ and $y \to u \in F$ imply $((x \to y) \to y) \to u \in F$, for every $x, y, u \in M$.

Proof (a) $\Rightarrow$ (b): Let F be a fantastic filter of M and $x \in M$. Since $0 \to x = 1 \in F$, we obtain from (5) that $x^{--} \to x = ((x \to 0) \to 0) \to x \in F$.

(b) $\Rightarrow$ (c): Suppose that F is a filter of M such that $x^{--} \to x \in F$ for every $x \in M$. Let $x, y, u \in M$, $x \to u \in F$ and $y \to u \in F$. Since $x \to u \leq u^- \to x^-$ and $y \to u \leq u^- \to y^-$, we get $u^- \to x^- \in F$ and $u^- \to y^- \in F$, and thus $(u^- \to x^-) \wedge (u^- \to y^-) \in F$.

Moreover,

$$\begin{aligned}&(u^- \to x^-) \wedge (u^- \to y^-) = u^- \to (x^- \wedge y^-)\\ &\quad = u^- \to (y^- \odot (y^- \to x^-)) = u^- \to (y^- \odot (y^- \to (x \to 0))\\ &\quad = u^- \to (y^- \odot (x \to (y^- \to 0)) = u^- \to (y^- \odot (x \to y^{--})).\end{aligned}$$

Further,

$$\begin{aligned}&(u^- \to (y^- \odot (x \to y^{--}))) \to (u^- \to (y^- \odot (x \to y)))\\ &\quad \geq (y^- \odot (x \to y^{--})) \to (y^- \odot (x \to y)))\\ &\quad \geq (x \to y^{--}) \to (x \to y) \geq y^{--} \to y \in F,\end{aligned}$$

therefore also $u^- \to (y^- \odot (x \to y)) \in F$.

Moreover,

$$u^- \to (y^- \odot (x \to y)) \leq (y^- \odot (x \to y))^- \to u^{--} = ((x \to y) \to y^{--}) \to u^{--},$$

hence $((x \to y) \to y^{--} \to u^{--} \in F$. Further we have

$$\begin{aligned}&(((x \to y) \to y^{--}) \to u^{--}) \to (((x \to y) \to y) \to u^{--})\\ &\quad \geq ((x \to y) \to y) \to ((x \to y) \to y^{--}) \geq y \to y^{--} = 1 \in F,\end{aligned}$$

thus $((x \to y) \to y) \to u^{--} \in F$.

Moreover,

$$(((x \to y) \to y) \to u^{--}) \to (((x \to y) \to y) \to u) \geq u^{--} \to u \in F,$$

therefore also $((x \to y) \to y) \to u \in F$.

(c) $\Rightarrow$ (a): If F satisfies the condition (c), then for $u = x$ we get that whether $y \to x \in F$ then $((x \to y) \to y) \to x \in F$, for every x, $y \in M$, hence F is a fantastic filter of M. $\square$

Theorem 4.5 *If F_1 and F_2 are filters of an $R\ell$-monoid M, $F_1 \subseteq F_2$ and F_1 is fantastic in M, then F_2 is also a fantastic filter of M.*

Proof Let F_1 and F_2 be filters of M, $F_1 \subseteq F_2$, and let F_1 be fantastic. Then by Theorem 4.4, $x^{--} \to x \in F_1 \subseteq F_2$, for every $x \in M$, hence F_2 is also fantastic. □

Theorem 4.6 *A filter F of an $R\ell$-monoid M is fantastic if and only if M/F is an MV-algebra.*

Proof Let F be a filter of M. Then F is fantastic if and only if $x^{--} \to x \in F$ for every $x \in M$, which is equivalent to the following conditions in M/F:

$$x^{--}/F \to x/F = F, \quad x^{--}/F \leq x/F \quad \text{and} \quad x^{--}/F = x/F,$$

for every $x/F \in M/F$, and this is equivalent to M/F is an MV-algebra. □

Proposition 4.7 *If F is a maximal filter of an $R\ell$-monoid M, then F is fantastic.*

Proof It follows from [3, Proposition 3.5], where it is proved that M/F is an MV-algebra for every maximal filter F of M. □

Remark 4.8 The *MV-filters* of $R\ell$-monoids, i.e. filters such that the corresponding quotient $R\ell$-monoids are MV-algebras, were investigated in [16], [17] and [3]. By Theorem 4.6, MV-filters of $R\ell$-monoids are exactly their fantastic filters. If M is an $R\ell$-monoid, denote by $D(M) := \{x \in M : x^{--} = 1\}$ the set of all dense elements in M. Then $D(M)$ is a proper filter of M and a filter F of M is an MV-filter if and only if $D(M) \subseteq F$. Therefore we get as a consequence the following proposition.

Proposition 4.9 *A filter F of an $R\ell$-monoid M is fantastic if and only if $D(M) \subseteq F$.*

Proposition 4.10 *Let M be an $R\ell$-monoid. Then the following conditions are equivalent:*

(1) M is an MV-algebra.

(2) Every filter of M is fantastic.

(3) $\{1\}$ is a fantastic filter of M.

Proof (1) ⇒ (2): Let M be an MV-algebra and F be a filter of M. Since the class of MV-algebras is a subvariety of the variety of $R\ell$-monoids, the quotient $R\ell$-monoid M/F is also an MV-algebra. Therefore by Theorem 4.6, F is a fantastic filter.

(2) ⇒ (3): It is obvious.

(3) ⇒ (1): Let $\{1\}$ be a fantastic filter of M. Then $M \cong M/\{1\}$ is an MV-algebra. □

Theorem 4.11 *If F is a filter of an $R\ell$-monoid M, then the following conditions are equivalent.*

(a) F is a Boolean filter.

(b) F is an implicative and fantastic filter.

Proof By Proposition 3.14, a filter F is Boolean if and only if M/F is a Boolean algebra. Moreover, an $R\ell$-monoid M/F is a Boolean algebra if and only if M/F is an MV-algebra and $(x/F)\odot(x/F) = x/F$ for every $x/F \in M/F$. This is equivalent to $(x/F)^{--} = x/F$ and $(x/F)\odot(x/F) = x/F$, and it holds, by Theorems 4.6 and 3.4, if and only if F is a fantastic and implicative filter of M. □

We have characterized filters of $R\ell$-monoids such that the corresponding quotient $R\ell$-monoids are Heyting algebras, Boolean algebras and MV-algebras, respectively. (See e.g. Theorem 3.4, Proposition 3.14 and Theorem 4.6.) Now we will complete it for the case when the quotient $R\ell$-monoid is a BL-algebra.

A filter F of an $R\ell$-monoid M is called a *BL-filter* of M if

$$(x \to y) \vee (y \to x) \in F,$$

for every $x, y \in M$.

Theorem 4.12 *A filter F of an $R\ell$-monoid M is a BL-filter of M if and only if M/F is a BL-algebra.*

Proof We know that an $R\ell$-monoid is a BL-algebra if and only if it satisfies the identity of pre-linearity.

Let M be an $R\ell$-monoid and F be a filter of M. If $x, y \in M$, then

$$(x/F \to y/F) \vee (y/F \to x/F) = ((x \to y) \vee (y \to x))/F.$$

Hence $(x/F \to y/F) \vee (y/F \to x/F) = F$ if and only if $(x \to y) \vee (y \to x) \in F$. □

References

[1] Balbes, R., Dwinger, P.: Distributive Lattices. *Univ. of Missouri Press, Columbia, Missouri,* 1974.

[2] Cignoli, R., D'Ottaviano, I. M. L., Mundici, D.: Algebraic Foundations of Many-valued Reasoning. *Kluwer Acad. Publ., Dordrecht,* 2000.

[3] Dvurečenskij, A., Rachůnek, J.: *Probabilistic averaging in bounded commutative residuated ℓ-monoids.* Discrete Mathematics **306** (2006), 1317–1326.

[4] Font, J. M., Rodriguez, A. J., Torrens, A.: *Wajsberg algebras.* Stochastica **8** (1984), 5–31.

[5] Hájek, P.: Metamathematics of Fuzzy Logic. *Kluwer Acad. Publ., Dordrecht,* 1998.

[6] Hájek, P.: *Basic fuzzy logic and BL-algebras.* Soft Comput. **2** (1998), 124–128.

[7] Haveshki, M., Saeid, A. B., Eslami, E.: *Some types of filters in BL-algebras.* Soft Comput **10** (2006), 657–664.

[8] Iorgulescu, A.: *Classes of BCK algebras – Part I.* Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint nr. 1/2004, 1–33.

[9] Iorgulescu, A.: *Classes of BCK algebras – Part III.* Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint nr. 3/2004, 1–37.

[10] Jipsen, P., Tsinakis, C.: *A survey of residuated lattices.* In: J. Martinez (ed.): Ordered algebraic structures. *Kluwer Acad. Publ. Dordrecht,* 2002, 19–56.

[11] Kondo, M., Dudek, W. A.: *Filter theory of BL-algebras.* Soft Comput. **12** (2008), 419–423.

[12] Rachůnek, J.: *MV-algebras are categorically equivalent to a class of $DR\ell_{1(i)}$-semigroups.* Math. Bohemica **123** (1998), 437–441.

[13] Rachůnek, J.: *A duality between algebras of basic logic and bounded representable $DR\ell$-monoids.* Math. Bohemica **126** (2001), 561–569.

[14] Rachůnek, J., Šalounová, D.: *Boolean deductive systems of bounded commutative residuated ℓ-monoids.* Contrib. Gen. Algebra **16** (2005), 199–208.

[15] Rachůnek, J., Šalounová, D.: *Local bounded commutative residuated ℓ-monoids.* Czechoslovak Math. J. **57** (2007), 395–406.

[16] Rachůnek, J., Slezák, V.: *Negation in bounded commutative $DR\ell$-monoids.* Czechoslovak Math. J. **56** (2006), 755–763.

[17] Rachůnek, J., Slezák, V.: *Bounded dually residuated lattice ordered monoids as a generalization of fuzzy structures.* Math. Slovaca **56** (2006), 223–233.

[18] Swamy, K. L. N.: *Dually residuated lattice ordered semigroups III.* Math. Ann. **167** (1966), 71–74.

[19] Turunen, E.: *Boolean deductive systems of BL-algebras.* Arch. Math. Logic **40** (2001), 467-473.