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Estimation of the First Order Parameters in the Twoepoch Linear Model

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Abstract

The linear regression model, where the mean value parameters are divided into stable and nonstable part in each of both epochs of measurement, is considered in this paper. Then, equivalent formulas of the best linear unbiased estimators of this parameters in both epochs using partitioned matrix inverse are derived.

Key words: Twoepoch regression model; best linear unbiased estimators of the first order parameters.

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1 Motivation

Many real problems, convenient for linear statistical modelling, typically from the field of geodesy, are investigated in more than one epoch to obtain better estimations of the unknown mean value parameters (see [5], [6], [7] and also [3], [4]). From the principle of concrete situation we suppose that some mean value (first order) parameters does not change their values during epochs (i.e. *stable* parameters) contrary to so called *nonstable* parameters. There are also many other problems, convenient for linear modelling, where the role of stable and nonstable parameters is not exactly given. Dividing the first order parameter to the stable and nonstable part we achieve better fit of the corresponding linear model to the concrete situation. As an example we choose the problem from [1], p. 90. A dependence of petrol consumption in litres to the rate in kilometers per hour by a certain car marque was investigated. The quadratical trend was chosen as the most convenient to describe the dependence. Let us have a new car and an older car. Then it is clear that the petrol consumption will be generally greater by the old car but the quadratical dependence to the rate by both cars will stay approximately unchanged. If we adopt this situation as an example of twoepoch measurement (the most occured in simpler problems), we can select the linear and quadratical term parameters to be stable and absolute term parameter to be nonstable first order parameters. Estimation of the stable and nonstable parameters using both epochs together give us better information about the dependence and increasing consumption than estimations in single epochs separately.

Let us formalize the performed considerations. The results of the measurement could be described as

$$Y_{1i} = \beta_1 x_{1i}^2 + \beta_2 x_{1i} + \gamma_1 + \varepsilon_{1i}, \quad i = 1, \dots, n_1$$

in the first epoch and

$$Y_{2i} = \beta_1 x_{2i}^2 + \beta_2 x_{2i} + \gamma_2 + \varepsilon_{2i}, \quad i = 1, \dots, n_2$$

in the second epoch of measurement. Let us consider the $n_1 + n_2$ dimensional observation vector $\mathbf{Y} = (\mathbf{Y}_1', \mathbf{Y}_2')'$ after the second epoch of the measurent. The model described above could be generally rewritten in the form

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \gamma_1 \\ \gamma_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}. \tag{1}$$

The (design) matrices $\mathbf{X}_1, \mathbf{X}_2, \mathbf{W}_1, \mathbf{W}_2$ are known; $\boldsymbol{\beta} \in \mathbb{R}^r$ is a vector of the useful stable parameters, the same in both epochs; $(\boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2')' \in \mathbb{R}^{s_1+s_2}$ is a vector of nonstable parameters in the first and the second epoch of measurement.

With respect to formerly mentioned, let us consider the linear model (1), called the twoepoch model with the stable and nonstable parameters. We suppose that

1. $\mathrm{E}(\mathbf{Y}_1) = \mathbf{X}_1\boldsymbol{\beta} + \mathbf{W}_1\boldsymbol{\gamma}_1$, $\mathrm{E}(\mathbf{Y}_2) = \mathbf{X}_2\boldsymbol{\beta} + \mathbf{W}_2\boldsymbol{\gamma}_2$,
   $\forall\boldsymbol{\beta} \in \mathbb{R}^r, \forall\boldsymbol{\gamma}_1 \in \mathbb{R}^{s_1}, \forall\boldsymbol{\gamma}_2 \in \mathbb{R}^{s_2}$;
2. $\mathrm{var}\left[\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}\right] = \begin{pmatrix} \sigma^2\boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \sigma^2\boldsymbol{\Sigma}_2 \end{pmatrix}$, $\sigma^2 > 0$ is unknown parameter;
3. the matrix $\boldsymbol{\Sigma}_i$ is not a function of the vector $(\boldsymbol{\beta}', \boldsymbol{\gamma}_i')'$ for $i = 1, 2$.

If the matrix

$$\begin{pmatrix} \sigma^2\boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \sigma^2\boldsymbol{\Sigma}_2 \end{pmatrix}$$

is positive definite (p.d.) and rank

$$r\left[\begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 \end{pmatrix}\right] = r + s_1 + s_2 < n_1 + n_2,$$

the model is said to be *regular* (see [5], p. 13).

The mentioned problem produces

$$\mathbf{X}_1 = \begin{pmatrix} x_{11}^2 & x_{11} \\ \vdots & \vdots \\ x_{1n_1}^2 & x_{1n_1} \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} x_{21}^2 & x_{21} \\ \vdots & \vdots \\ x_{2n_2}^2 & x_{2n_2} \end{pmatrix},$$

$$\mathbf{W}_1 = (\overbrace{1, \ldots, 1}^{n_1-\text{times}})' = \mathbf{1}_{n_1}, \quad \mathbf{W}_2 = (\overbrace{1, \ldots, 1}^{n_2-\text{times}})' = \mathbf{1}_{n_2},$$

$$\boldsymbol{\beta} = (\beta_1, \beta_2)', \ \boldsymbol{\gamma}_1 = \gamma_1, \ \boldsymbol{\gamma}_2 = \gamma_2.$$

In the case of positive definitness of the matrices $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$ is the model evidently regular.

2 Notation and auxiliary statements

Let us summarize the notation, used throughout the paper:

$\mathbb{R}^n$	the space of all n-dimensional real vectors;
$\mathbf{u}$, $\mathbf{A}$	the real column vector, the real matrix;
$\mathbf{A}', r(\mathbf{A})$	the transpose, the rank of the matrix $\mathbf{A}$;
$\mathcal{M}(\mathbf{A})$, $\mathrm{Ker}(\mathbf{A})$	the range, the null space of the matrix $\mathbf{A}$;
$\mathbf{P}_A$	the orthogonal projector onto $\mathcal{M}(\mathbf{A})$ (in Euclidean sense);
$\mathbf{M}_A = \mathbf{I} - \mathbf{P}_A$	the orthogonal projector onto $\mathcal{M}^{\perp}(\mathbf{A}) = \mathrm{Ker}(\mathbf{A}')$;
$\mathbf{I}_k$	the $k \times k$ identity matrix;
$\mathbf{0}_{m,n}$	the $m \times n$ null matrix;
$\mathbf{1}_k$	$= (1, \ldots, 1)' \in \mathbb{R}^k$;
$F_{m,n}$	random variable with F distribution with m and n degrees of freedom;
$F_{m,n}(1-\alpha)$	$(1-\alpha)$-quantile of this distribution.

Lemma 1 *Inverse of partitioned p.d. matrix*

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{D} \\ \mathbf{B}' & \mathbf{C} & \mathbf{0} \\ \mathbf{D}' & \mathbf{0} & \mathbf{E} \end{pmatrix} \quad \textit{is equal to} \quad \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} & \mathbf{Q}_{13} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix},$$

where ($\mathbf{Q}_{21} = (\mathbf{Q}_{12})'$, $\mathbf{Q}_{31} = (\mathbf{Q}_{13})'$, $\mathbf{Q}_{32} = (\mathbf{Q}_{23})'$*)*

$$\begin{pmatrix} \mathbf{Q}_{11} & -\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1} & -\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1} \\ -\mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{11} & \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1} & -\mathbf{Q}_{21}\mathbf{D}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{11} & -\mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{12} & \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1} \end{pmatrix},$$

with

$$\mathbf{Q}_{11} = (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}' - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}$$

(Version I); equivalently

$$\begin{aligned}
\mathbf{Q}_{11} &= (\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1} + (\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}\mathbf{B}\mathbf{Q}_{22}\mathbf{B}'(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1},\\
\mathbf{Q}_{12} &= -(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}\mathbf{B}\mathbf{Q}_{22},\\
\mathbf{Q}_{13} &= -\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1},\\
\mathbf{Q}_{22} &= [\mathbf{C} - \mathbf{B}'(\mathbf{A} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}')^{-1}\mathbf{B}]^{-1},\\
\mathbf{Q}_{23} &= -\mathbf{Q}_{21}\mathbf{D}\mathbf{E}^{-1},\\
\mathbf{Q}_{33} &= \mathbf{E}^{-1} + \mathbf{E}^{-1}\mathbf{D}'\mathbf{Q}_{11}\mathbf{D}\mathbf{E}^{-1},
\end{aligned}$$

(Version II) and equivalently

$$\begin{aligned}
\mathbf{Q}_{11} &= (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1} + (\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{D}\mathbf{Q}_{33}\mathbf{D}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1},\\
\mathbf{Q}_{12} &= -\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1},\\
\mathbf{Q}_{13} &= -(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{D}\mathbf{Q}_{33},\\
\mathbf{Q}_{22} &= \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{11}\mathbf{B}\mathbf{C}^{-1},\\
\mathbf{Q}_{23} &= -\mathbf{C}^{-1}\mathbf{B}'\mathbf{Q}_{13},\\
\mathbf{Q}_{33} &= [\mathbf{E} - \mathbf{D}'(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{D}]^{-1},
\end{aligned}$$

(Version III).

Proof The statement can be proved directly using [3, Theorem 1 and Remark 1–3], with proper Rohde formula (see [2, Theorem 8.5.11, p. 99]). □

It can be easily shown that there exist five versions of such inverse alltogether but only above mentioned are convenient for our later purposes.

Lemma 2 *Let us consider regular linear model*

$$(\mathbf{Y},\, \mathbf{X}\boldsymbol{\delta}, \sigma^2\mathbf{V}),\ \ \boldsymbol{\delta} \in \mathbb{R}^k,\, \sigma^2 > 0 \tag{2}$$

where $\mathbf{Y}$ *is* n*-dimensional normally distributed observation vector,* $\mathbf{X}$ $n \times k$ *design matrix* ($r(\mathbf{X}) = k < n$) *and* $\boldsymbol{\Sigma}$ *is known p.d. variance matrix of the type* $n \times n$*. Then the best linear unbiased estimator (BLUE) of the vector* $\boldsymbol{\delta}$ *equals*

$$\widehat{\boldsymbol{\delta}} = (\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \tag{3}$$

with the variance matrix

$$\operatorname{var}(\widehat{\boldsymbol{\delta}}) = \sigma^2(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}. \tag{4}$$

Unbiased and invariant estimator of σ^2 *is*

$$\widehat{\sigma^2} = (\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\delta}})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y} - \mathbf{X}\widehat{\boldsymbol{\delta}})/(n-k). \tag{5}$$

Let null hypothesis about parameter $\boldsymbol{\delta}$ *is*

$$H_0 : \mathbf{H}\boldsymbol{\delta} + \mathbf{h} = \mathbf{0}, \tag{6}$$

where $\mathbf{H}$ *is* $q \times k$ *matrix with* $r(\mathbf{H}) = q < k$*, and alternative hypothesis*

$$H_a : \mathbf{H}\boldsymbol{\delta} + \mathbf{h} \neq \mathbf{0}. \tag{7}$$

Then, in premise of validity of H_0*, the random variable* F

$$F = \frac{(\mathbf{H}\widehat{\boldsymbol{\delta}} + \mathbf{h})'[\mathbf{H}(\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{H}']^{-1}(\mathbf{H}\widehat{\boldsymbol{\delta}} + \mathbf{h})}{\widehat{\sigma^2}} \tag{8}$$

is $F_{q,n-k}$ *distributed.*

Proof see [5, p. 13, Theorem 1.1.1 and p. 54, Theorem 1.8.9]. □

3 Best linear unbiased estimators

Let the twoepoch linear regression model (1) be given. This model arises by sequential realizations of the linear partial regression models,

$$\mathbf{Y}_1 = (\mathbf{X}_1, \mathbf{W}_1)\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma}_1 \end{pmatrix} + \boldsymbol{\varepsilon}_1,\ \mathrm{var}(\mathbf{Y}_1) = \sigma^2\boldsymbol{\Sigma}_1 \tag{9}$$

and

$$\mathbf{Y}_2 = (\mathbf{X}_2, \mathbf{W}_2)\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma}_2 \end{pmatrix} + \boldsymbol{\varepsilon}_2,\ \mathrm{var}(\mathbf{Y}_2) = \sigma^2\boldsymbol{\Sigma}_2, \tag{10}$$

representing the model of the measurement within the first and second epoch, respectively. Let us remark that the parameter σ^2 is supposed to be the same in both epochs. Althougt this condition could be too restricting in some cases we adopt it to make the computations easily. Moreover there are many situations, mainly in simpler problems, where this condition is acceptable. The further derived formulas are more complicated that in general case but they show the structure of the twoepoch model what is useful in many applications. In addition, thanks to Lemma 1 the formulas in the twoepoch model will be derived in a friendly methodical way. The next theorem can be used to verify the stableness of the first order parameters.

Theorem 1 *The BLUE of the parameters* $\boldsymbol{\beta}$, $\boldsymbol{\gamma}_1$ *and* $\boldsymbol{\gamma}_2$ *in the single first and second epoch (models (9) and (10)) are for* $i = 1, 2$

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{W_i}^{\Sigma_i^{-1}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{W_i}^{\Sigma_i^{-1}}\mathbf{Y}_i, \\
\widehat{\boldsymbol{\gamma}_i}^{(i)} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}(\mathbf{Y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}^{(i)}),
\end{aligned}$$

equivalently

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}}^{(i)} &= (\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{X}_i)^{-1}\mathbf{X}_i'\boldsymbol{\Sigma}_i^{-1}(\mathbf{Y}_i - \mathbf{W}_i\widehat{\boldsymbol{\gamma}}_i^{(i)}),\\
\widehat{\boldsymbol{\gamma}}_i^{(i)} &= (\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{X_i}^{\Sigma_i^{-1}}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{X_i}^{\Sigma_i^{-1}}\mathbf{Y}_i,
\end{aligned}
$$

where $\boldsymbol{\Sigma}_i^{-1}\mathbf{M}_{W_i}^{\Sigma_i^{-1}} = \boldsymbol{\Sigma}_i^{-1} - \boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i(\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}\mathbf{W}_i)^{-1}\mathbf{W}_i'\boldsymbol{\Sigma}_i^{-1}$.

Proof see [5, p. 369, Theorem 9.1.2]. □

Theorem 2 *In the regular twoepoch linear model (1), the BLUE of the parameters* $\boldsymbol{\beta}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ *in both epochs equals*

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}} &= (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{W_1}^{\Sigma_1^{-1}}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{W_2}^{\Sigma_2^{-1}}\mathbf{X}_2)^{-1}\\
&\quad\times(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{W_1}^{\Sigma_1^{-1}}\mathbf{Y}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{W_2}^{\Sigma_2^{-1}}\mathbf{Y}_2),\\
\widehat{\boldsymbol{\gamma}_1} &= (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}(\mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}}),\\
\widehat{\boldsymbol{\gamma}_2} &= (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}(\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}),
\end{aligned}
$$

(Version I); equivalently

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}} &= (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{W_2}^{\Sigma_2^{-1}}\mathbf{X}_2)^{-1}\\
&\quad\times[\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}(\mathbf{Y}_1 - \mathbf{W}_1\widehat{\boldsymbol{\gamma}_1}) + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{W_2}^{\Sigma_2^{-1}}\mathbf{Y}_2],\\
\widehat{\boldsymbol{\gamma}_1} &= [\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{W}_1 - \mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{W_2}^{\Sigma_2^{-1}}\mathbf{X}_2)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{W}_1]^{-1}\\
&\quad\times[\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 - \mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{W_2}^{\Sigma_2^{-1}}\mathbf{X}_2)^{-1}\\
&\quad\times(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{M}_{W_2}^{\Sigma_2^{-1}}\mathbf{Y}_2)],\\
\widehat{\boldsymbol{\gamma}_2} &= (\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{W}_2)^{-1}\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}(\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}),
\end{aligned}
$$

(Version II); equivalently

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}} &= (\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{W_1}^{\Sigma_1^{-1}}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-1}\\
&\quad\times[\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{W_1}^{\Sigma_1^{-1}}\mathbf{Y}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}(\mathbf{Y}_2 - \mathbf{W}_2\widehat{\boldsymbol{\gamma}_2})],\\
\widehat{\boldsymbol{\gamma}_1} &= (\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{W}_1)^{-1}\mathbf{W}_1'\boldsymbol{\Sigma}_1^{-1}(\mathbf{Y}_1 - \mathbf{X}_1\widehat{\boldsymbol{\beta}}),\\
\widehat{\boldsymbol{\gamma}_2} &= [\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{W}_2 - \mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{W_1}^{\Sigma_1^{-1}}\mathbf{X}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{W}_2]^{-1}\\
&\quad\times[\mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2 - \mathbf{W}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2 + \mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{W_1}^{\Sigma_1^{-1}}\mathbf{X}_1)^{-1}\\
&\quad\times(\mathbf{X}_1'\boldsymbol{\Sigma}_1^{-1}\mathbf{M}_{W_1}^{\Sigma_1^{-1}}\mathbf{Y}_1 + \mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2)],
\end{aligned}
$$

(Version III).

Proof Lemma 2 is ready to help us in proving this theorem. Here

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{W}_1 & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2 \end{pmatrix}, \; \mathbf{\Sigma} = \begin{pmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 \end{pmatrix}, \; \boldsymbol{\delta} = (\boldsymbol{\beta}', \boldsymbol{\gamma}_1', \boldsymbol{\gamma}_2')',$$

so that

$$\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X} = \begin{pmatrix} \mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{X}_1 + \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{X}_2 & \mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{W}_1 & \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{W}_2 \\ \mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{X}_1 & \mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{W}_1 & \mathbf{0} \\ \mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{X}_2 & \mathbf{0} & \mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{W}_2 \end{pmatrix}. \tag{11}$$

Using Lemma 1, Versions I–III, and obvious relations

$$\begin{aligned}
\widehat{\boldsymbol{\beta}} &= \mathbf{Q}_{11}(\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Y}_2) + \mathbf{Q}_{12}\mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{Q}_{13}\mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Y}_2, \\
\widehat{\boldsymbol{\gamma}_1} &= \mathbf{Q}_{21}(\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Y}_2) + \mathbf{Q}_{22}\mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{Q}_{23}\mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Y}_2, \\
\widehat{\boldsymbol{\gamma}_2} &= \mathbf{Q}_{31}(\mathbf{X}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{X}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Y}_2) + \mathbf{Q}_{32}\mathbf{W}_1'\mathbf{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{Q}_{33}\mathbf{W}_2'\mathbf{\Sigma}_2^{-1}\mathbf{Y}_2
\end{aligned}$$

we get the results—Versions I–III (in this order). □

Let us remark that functional dependence of above derived estimators to the other estimator(s) coheres with the functional dependence of diagonal blocks of inverse matrices in Lemma 1 to their other diagonal block(s). So we obtained such estimators of each of the parameters $\boldsymbol{\beta}, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$ that are not a function of any other parameter's estimator. For instance, in Version I the estimator $\boldsymbol{\beta}$ is not a function of $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$, in Version II $\boldsymbol{\gamma}_1$ is not a function of $\boldsymbol{\beta}, \boldsymbol{\gamma}_2$ and analogously in Version III. This result is important mainly from the theoretic point of view and as an effective tool for checking of numerical results. In the practice, Version I seems to be the most convenient for computing the estimators.

Example 1 Let us test the hypothesis $H_0 : \boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2$ in the regular model (1) with $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{X}$, $\mathbf{W}_1 = \mathbf{W}_2 = \mathbf{W}$ and $\mathbf{\Sigma}_1 = \mathbf{\Sigma}_2 = \mathbf{\Sigma}$ and without evaluating the corresponding estimators of the first order parameters. We will follow Lemma 2. Here $\mathbf{H} = (\mathbf{0}, \mathbf{I}, -\mathbf{I})$, so the F statistics (8) equals

$$\frac{(\widehat{\boldsymbol{\gamma}_1} - \widehat{\boldsymbol{\gamma}_2})'(\mathbf{Q}_{22} - \mathbf{Q}_{23} - \mathbf{Q}_{32} + \mathbf{Q}_{33})^{-1}(\widehat{\boldsymbol{\gamma}_1} - \widehat{\boldsymbol{\gamma}_2})}{\widehat{\sigma^2}},$$

where $\mathbf{Q}_{22}, \mathbf{Q}_{23}, \mathbf{Q}_{32}, \mathbf{Q}_{33}$ are given by inverse of (11) using Lemma 1. Then $\widehat{\boldsymbol{\gamma}_1} - \widehat{\boldsymbol{\gamma}_2}$ using Version I from Theorem 1 for $\widehat{\boldsymbol{\gamma}_1}$ and $\widehat{\boldsymbol{\gamma}_2}$ is of the form

$$(\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{W})^{-1}\mathbf{W}'\mathbf{\Sigma}^{-1}(\mathbf{Y}_1 - \mathbf{Y}_2).$$

The same term $\widehat{\boldsymbol{\gamma}_1} - \widehat{\boldsymbol{\gamma}_2}$ with $\widehat{\boldsymbol{\gamma}_1}$ from Version II and $\widehat{\boldsymbol{\gamma}_2}$ from Version III is

$$\begin{aligned}
&[\mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{W} - \mathbf{W}'\mathbf{\Sigma}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{M}_W^{\Sigma^{-1}}\mathbf{X} + \mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{W}]^{-1} \\
&\times\mathbf{W}'\mathbf{\Sigma}^{-1}[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{M}_W^{\Sigma^{-1}}\mathbf{X} + \mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{P}_W^{\Sigma^{-1}}](\mathbf{Y}_1 - \mathbf{Y}_2).
\end{aligned}$$

Moreover $s_1 = s_2 = s$ and $n_1 = n_2 = N$, so we reject H_0 on the level α, if $F \geq F_{s,2N-r-2s}(1-\alpha)$.

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