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# Conjugated Algebras*

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**Abstract**

We generalize the correspondence between basic algebras and lattices with section antitone involutions to a more general case where no lattice properties are assumed. These algebras are called conjugated if this correspondence is one-to-one. We get conditions for the conjugary of such algebras and introduce the induced relation. Necessary and sufficient conditions are given to indicated when the induced relation is a quasiorder which has "nice properties", e.g. the unary operations are antitone involutions on the corresponding intervals.

**Key words:** Conjugated alegebras, basic algebra, section antitone involution, quasiorder.

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Correspondence between MV-algebras and bounded distributive lattices with section antitone involutions is well-known, see e.g. [3] and [5]. This was generalized for basic algebras and general bounded lattices with section antitone involutions, see [2] and [3] for details. Semilattices and lattices with section antitone involutions were treated separately in [1]. If a bounded lattice is replaced by the so-called $\lambda$-lattice, the corresponding algebra is called an NMV-algebra, an non-associative generalization of an MV-algebra, see [4]. If a little less is assumed, we get the correspondence between weak basic algebras and directoids with section antitone involutions, see [6]. These attempts motivate us to find a general correspondence between algebras of two sorts. One of them are "MV-like algebras", the other are "semilattice-like algebras" with a set of unary operations. Since in all the aforementioned cases the "semilattice-like algebras" were ordered, we add an assumption that our algebras of the second sort will

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be at least quasiordered. If there is a one-to-one correspondence between these algebras, we will say that they are conjugated.

At first, we get precise meaning to mentioned concepts.

We consider two kinds of algebras. The first are algebras $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$. For the sake of brevity, we will denote $1 := \neg 0$ the algebraic constant of $\mathcal{A}$.

The second are algebras $\mathcal{L} = (A; \sqcup, ({}^b)_{b \in A}, 0)$ where $\sqcup$ is a binary operation, 0 is a nullary operation and for each $b \in A$, ${}^b$ is a unary operation on $A$, i.e. it is a mapping $A \to A$ assigning to $x \in A$ an element $x^b$. Denote by $1 := 0^0$. To every $\mathcal{A} = (A; \oplus, \neg, 0)$ there can be assigned an algebra $\mathcal{L}(A) = (A; \sqcup, ({}^b)_{b \in A}, 0)$, where

$$x \sqcup y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x^y = \neg x \oplus y.$$

To every $\mathcal{L} = (L; \sqcup, ({}^b)_{b \in L}, 0)$ there can be assigned an algebra $\mathcal{A}(L) = (L; \oplus, \neg, 0)$, where

$$x \oplus y = (x^0 \sqcup y)^y \quad \text{and} \quad \neg x = x^0.$$

We call algebras $\mathcal{A} = (A; \oplus, \neg, 0)$ and $\mathcal{L} = (L; \sqcup, ({}^b)_{b \in A}, 0)$ *conjugated* if

$$\mathcal{L} = \mathcal{L}(A) \quad \text{and} \quad \mathcal{A} = \mathcal{A}(L).$$

This yields $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$ and $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$, i.e. if they share the same base-set and the aforementioned assignments are one-to-one correspondences.

At first, we can describe the following properties of conjugated algebras.

**Theorem 1** *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfy the conditions*

*(A1)* $\neg\neg x = x$;

*(A2)* $x \oplus 0 = x$;

*(A3)* $\neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y$.

*Then $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$ and $\mathcal{L}(A)$ satisfies the conditions*

*(L1)* $(x \sqcup y)^{yy} = x \sqcup y$;

*(L2)* $x^y = (x \sqcup y)^y$;

*(L3)* $x \sqcup 0 = x$.

**Proof** Assume that $\mathcal{A}$ satisfies (A1), (A2) and (A3) and denote by $\boxplus, \sim$ the operations of $\mathcal{A}(\mathcal{L}(A))$. Of course, the nullary operation 0 is the same both in $\mathcal{A}$ and $\mathcal{A}(\mathcal{L}(A))$. We have by (A2)

$$\sim x = x^0 = \neg x \oplus 0 = \neg x.$$

Further, we compute by (A1) and (A3)

$$\begin{aligned} x \boxplus y = (x^0 \sqcup y)^y = (\neg x \sqcup y)^y = (\neg(\neg\neg x \oplus y) \oplus y)^y \\ = \neg(\neg(x \oplus y) \oplus y) \oplus y = x \oplus y \end{aligned}$$

thus $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$.

Further, applying (A3), we conclude

$$x^y = \neg x \oplus y = \neg(\neg(\neg x \oplus y) \oplus y) \oplus y = \neg(x \sqcup y) \oplus y = (x \sqcup y)^y$$

proving (L2). Using this we obtain

$$(x \sqcup y)^{yy} = x^{yy} = \neg(\neg x \oplus y) \oplus y = x \sqcup y$$

which is (L1). Using (A1) and (A2) we prove also (L3):

$$x \sqcup 0 = \neg(\neg x \oplus 0) \oplus 0 = \neg\neg x = x. \qquad \square$$

**Theorem 2** *Let $\mathcal{L} = (L; \sqcup, (^b)_{b \in L}, 0)$ satisfy (L1), (L2) and (L3). Then $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$ and $\mathcal{A}(L)$ satisfies (A1), (A2) and (A3).*

**Proof** Assume that $\mathcal{L}$ satisfies (L1), (L2) and (L3) and denote by $\vee$ the binary operation and by $(f_b)_{b \in L}$ the set of unary operations of $\mathcal{L}(\mathcal{A}(L))$. Of course, the nullary operation 0 is the same in both the algebras. Then, by (L1),

$$x \vee y = \neg(\neg x \oplus y) \oplus y = (\neg x \oplus y)^y = (x \sqcup y)^{yy} = x \sqcup y.$$

Further, $\neg\neg x = x^{00} = (x \sqcup 0)^{00} = x \sqcup 0 = x$ by (L1), (L2) and (L3). Next, by (L2),

$$f_y(x) = \neg x \oplus y = ((\neg x)^0 \sqcup y)^y = (x^{00} \sqcup y)^y = (x \sqcup y)^y = x^y$$

thus $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$ and $\mathcal{A}(L)$ satisfies (A1). Analogously,

$$x \oplus 0 = (x^0 \sqcup 0)^0 = x^{00} = (x \sqcup 0)^{00} = x \sqcup 0 = x$$

thus $\mathcal{A}(L)$ satisfies (A2). Since $\mathcal{A}(L)$ already satisfies (A1), we can easily compute

$$\neg(\neg(x \oplus y) \oplus y) \oplus y = (\neg x \sqcup y)^y = (\neg x)^y = \neg\neg x \oplus y = x \oplus y$$

proving (A3). $\square$

**Corollary 1** *Let $\mathcal{A}$ satisfy (A1), (A2) and (A3). Then $\mathcal{A}$ and $\mathcal{L}(A)$ are conjugated. Let $\mathcal{L}$ satisfy (L1), (L2) and (L3). Then $\mathcal{L}$ and $\mathcal{A}(L)$ are conjugated.*

**Corollary 2** *Let $\mathcal{A}, \mathcal{L}$ be conjugated algebras. Then $\mathcal{A}$ satisfies (A1), (A2), (A3) if and only if $\mathcal{L}$ satisfies (L1), (L2), (L3).*

**Remark 1** As mentioned in the introduction, the correspondence between $\mathcal{A} = (A; \oplus, \neg, 0)$ and $\mathcal{L} = (A; \sqcup, (^b)_{b \in A}, 0)$ was studied for several cases. The results are as follows:

(1) If $\mathcal{A}$ is a basic algebra then $\mathcal{L} = \mathcal{L}(A)$ is a bounded semilattice with section antitone involutions (SAI for short);

(2) If $\mathcal{A}$ is an MV-algebra then $\mathcal{L} = \mathcal{L}(A)$ is a bounded semilattice with SAI satisfying the Exchange Property;

(3) If $\mathcal{A}$ is an NMV-algebra then $\mathcal{L} = \mathcal{L}(A)$ is a commutative directoid with SAI;

(4) If $\mathcal{A}$ is a weak basic algebra then $\mathcal{L} = \mathcal{L}(A)$ is a directoid with SAI (not necessarily commutative).

In all the cases, $\mathcal{A}$ and $\mathcal{L}$ are conjugated and there exists an induced order such that 0 (or 1) is the least (or the greatest) element and $y \leq x \sqcup y$. We are going to study this question concerning some "order-like" relation also on conjugated algebras in general.

Define a binary relation $\leq$ on an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ as follows

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

Call $\leq$ the *induced relation* on $\mathcal{A}$.

Let us note that $1 = \neg 0$. If $\mathcal{A}$ satisfies (A1), then also $\neg 1 = \neg\neg 0 = 0$.

**Lemma 1** *The induced relation $\leq$ on $\mathcal{A}$ is reflexive if and only if $\mathcal{A}$ satisfies the identity*

*(P)* $\neg x \oplus x = 1$.

*Let $\mathcal{A}$ satisfy (A1). Then $0 \leq x \leq 1$ for each $x \in A$ if and only if $\mathcal{A}$ satisfies the identity*

*(A4)* $1 \oplus x = 1 = x \oplus 1$.

**Proof** The first assertion is trivial. For the second one, $0 \leq x$ is equivalent to $1 \oplus x = \neg 0 \oplus x = 1$ and $x \leq 1$ is equivalent to $\neg x \oplus 1 = 1$ for each $x \in A$, i.e. due to (A1), $\mathcal{A}$ satisfies also the identity $x \oplus 1 = 1$. □

**Lemma 2** *Let $\mathcal{A}, \mathcal{L}$ be conjugated algebras and $\leq$ be the induced relation on $\mathcal{A}$. Let $\mathcal{A}$ satisfy (A1) and (A4) and $\mathcal{L}$ satisfy (L1). Then the following conditions are equivalent*

*(a)* $1^x = x$ *and* $x^x = 1$;

*(b)* $x \leq y$ *if and only if* $x \sqcup y = y$.

**Proof** (a)⇒(b): Let $x \leq y$. Then

$$(x \sqcup y)^y = \neg x \oplus y = 1$$

thus, by (L1),

$$x \sqcup y = (x \sqcup y)^{yy} = 1^y = y.$$

Conversely, if $x \sqcup y = y$ then

$$\neg x \oplus y = (x \sqcup y)^y = y^y = 1,$$

i.e. $x \leq y$.

(b)⇒(a): Applying Lemma 1, (A4) yields $0 \leq x$ and, by the assumption (b), $0 \sqcup x = x$. By (A4) and (L1) we have

$$1^x = (1 \oplus x)^x = (\neg 1 \sqcup x)^{xx} = (0 \sqcup x)^{xx} = 0 \sqcup x = x.$$

Similarly,

$$x^x = (0 \sqcup x)^x = 0^0 \oplus x = 1 \oplus x = 1. \qquad \square$$

**Lemma 3** *Let $\mathcal{A}$ and $\mathcal{L}$ be conjugated algebras. Then $x \leq x \sqcup y$ if and only if $\mathcal{A}$ satisfies*

*(A5)* $\neg x \oplus (\neg(\neg x \oplus y) \oplus y) = 1$.

**Proof** By the definition of $\leq$ we have that

$$x \leq x \sqcup y \quad \text{if and only if} \quad \neg x \oplus (x \sqcup y) = 1.$$

However, $\mathcal{A}, \mathcal{L}$ are conjugated thus $x \sqcup y = \neg(\neg x \oplus y) \oplus y$. $\square$

A binary relation is called a *quasiorder* if it is reflexive and transitive. We are going to characterize algebras $\mathcal{A} = (A; \oplus, \neg, 0)$ for which the induced relation is a quasiorder which has a special meaning for the assigned algebra $\mathcal{L}$.

**Lemma 4** *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ satisfy the identities (A1) and*

*(A6)* $0 \oplus x = x$;

*(A7)* $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

*Then the induced relation $\leq$ is transitive.*

**Proof** Assume $x \leq y$ and $y \leq z$, i.e. $\neg x \oplus y = 1$ and $\neg y \oplus z = 1$. By (A1), (A7) and (A6) we compute

$$\begin{aligned} 1 &= \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus (\neg x \oplus z) \\ &= \neg(\neg(\neg 1 \oplus y) \oplus z) \oplus (\neg x \oplus z) = \neg(\neg(0 \oplus y) \oplus z) \oplus (\neg x \oplus z) \\ &= \neg(\neg y \oplus z) \oplus (\neg x \oplus z) = \neg 1 \oplus (\neg x \oplus z) = 0 \oplus (\neg x \oplus z) = \neg x \oplus z \end{aligned}$$

whence $x \leq z$. $\square$

Let $(A; \leq)$ be a quasiordered set and $f : A \to A$ be a mapping. We say that $f$ is *antitone* if $x \leq y$ yields $f(y) \leq f(x)$ and $f$ is an *involution* if $f(f(x)) = x$ for every $x \in A$. If $a, b \in A$ and $a \leq b$, by an interval $[a, b]$ is meant the subset of $A$ given by $[a, b] = \{x \in A; a \leq x \leq b\}$.

**Theorem 3** *Let $\mathcal{A}, \mathcal{L}$ be conjugated algebras, let $\leq$ be the induced relation on $\mathcal{A}$. Let $\mathcal{A}$ satisfy (A1), (A2), (A3), (A4) and (A6). The following conditions are equivalent*

*(1) $\mathcal{A}$ satisfies (A7);*

*(2)* $\leq$ *is a quasiorder on* $A$ *such that* $x \leq x \sqcup y$ *for each* $x, y \in A$ *and for each* $z \in A$ *the mapping* $x \mapsto x^z$ *is an antitone involution on the interval* $[z, 1]$.

**Proof** (1) $\Rightarrow$ (2): Put $y = 0 = z$ in (A7). We get $\neg x \oplus x = 1$ which is (P) of Lemma 1, i.e. $\leq$ is reflexive. Since $\mathcal{A}$ satisfies (A6) and (A7), $\leq$ is transitive by Lemma 4 and hence $(A; \leq)$ is a quasiordered set.

Assume $x \leq y$. Then $\neg x \oplus y = 1$ and, by (A7),

$$1 = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus (\neg x \oplus z) = \neg(\neg y \oplus z) \oplus (\neg x \oplus z)$$

thus

$$\neg y \oplus z \leq \neg x \oplus z. \qquad (*)$$

For $z = 0$ we have $x \leq y \Rightarrow \neg y \leq \neg x$ which is equivalent to

$$\neg x \oplus y = 1 \quad \Rightarrow \quad y \oplus \neg x = 1. \qquad (**)$$

Taking $z = 0$ and replacing $x$ by $\neg x$ in (A7), we obtain

$$(\neg(\neg x \oplus y) \oplus y) \oplus \neg x = 1$$

thus, by $(**)$, we obtain

$$\neg x \oplus (\neg(\neg x \oplus y) \oplus y) = 1$$

which yields

$$x \leq \neg(\neg x \oplus y) \oplus y = x \sqcup y.$$

Let $x, y \in [z, 1]$ and $x \leq y$. By $(*)$ we have $y^z = \neg y \oplus z \leq \neg x \oplus z = x^z$ thus the mapping $x \mapsto x^z$ is antitone. By (A4) we have $x^z \leq 1$. Applying $(*)$ twice and using (A1), we obtain

$$x \leq y \quad \Rightarrow \quad x \oplus z \leq y \oplus z. \qquad (***)$$

Since $\neg x \leq 1$ by (A4), $(*)$ yileds $0 \leq x$ thus, by $(***)$ and (A6), we obtain

$$y = 0 \oplus y \leq x \oplus y.$$

This yields $z \leq \neg x \oplus z = x^z$. We have shown that $x \mapsto x^z$ is really a mapping of the interval $[z, 1]$ into itself. By (L1) and (L2), it is an involution. We have shown (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): By (2) we have $\neg x \leq \neg x \sqcup y$ where the induced relation $\leq$ is a quasiorder on $A$. By (2),

$$\neg(\neg(x \oplus y) \oplus y) \oplus z) = \neg(\neg x \sqcup y) \oplus z =$$

$$= ((\neg x \sqcup y) \sqcup z)^z = (\neg x \sqcup y)^z \leq (\neg x)^z = (\neg x \sqcup z)^z = x \oplus z$$

thus $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$ which is just (A7). $\square$

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