

On the Measurability of Sets of Pairs of Intersecting Nonisotropic Straight Lines of Type Beta in the Simply Isotropic Space

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Abstract

The measurable sets of pairs of intersecting non-isotropic straight lines of type β and the corresponding densities with respect to the group of general similitudes and some its subgroups are described. Also some Crofton-type formulas are presented.

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1 Introduction

The *simply isotropic space* $I_3^{(1)}$ (see [8]) is defined as a projective space $\mathbb{P}_3(\mathbb{R})$ in which the absolute consists of a plane ω (the *absolute plane*) and two complex conjugate straight lines f_1, f_2 (the *absolute lines*) within ω . In homogeneous coordinates (x_0, x_1, x_2, x_3) we can choose the plane $x_0 = 0$ as the plane ω , the line $x_0 = 0, x_1 + ix_2 = 0$ as the line f_1 , and the line $x_0 = 0, x_1 - ix_2 = 0$ as the line f_2 . Then the intersecting point F of f_1 and f_2 , which is called an *absolute point*, has coordinates $(0, 0, 0, 1)$. All regular projectivities transforming the absolute figure into itself form the 8-parametric group G_8 of *general simply*

isotropic similitudes. In affine coordinates (x, y, z) with respect to the affine coordinate system $(O, \vec{e}_1, \vec{e}_2, \vec{e}_3)$, any similitude of G_8 can be written in the form ([8, p. 3])

$$\begin{aligned}\bar{x} &= c_1 + c_7(x \cos \varphi - y \sin \varphi), \\ \bar{y} &= c_2 + c_7(x \sin \varphi + y \cos \varphi), \\ \bar{z} &= c_3 + c_4x + c_5y + c_6z,\end{aligned}\tag{1}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$, and φ are real parameters and $c_7 > 0$.

A plane in $I_3^{(1)}$ is said to be *non-isotropic* if its infinite line is not incident with the absolute point F ; otherwise the plane is called *isotropic*.

A straight line in $I_3^{(1)}$ is said to be (*completely*) *isotropic* if its infinite point coincides with the absolute point F ; otherwise the straight line is said to be *non-isotropic* ([8, p. 5]).

Let G_1 and G_2 be two non-isotropic straight lines and let us denote by U_1 and U_2 their infinite points, respectively. The straight lines G_1 and G_2 are said to be of *type β* if the points U_1, U_2 , and F are collinear; otherwise the straight lines are said to be of *type α* ([8, p. 45]).

We will consider also the following subgroups of G_8 :

I. $B_7 \subset G_8 \iff c_7 = 1$. This is the group of simply isotropic similitudes of the δ -distance ([8, p. 5]).

II. $S_7 \subset G_8 \iff c_6 = 1$. This is the group of simply isotropic similitudes of the s -distance ([8, p. 6]).

III. $W_7 \subset G_8 \iff c_6 = c_7$. This is the group of simply isotropic *angular* similitudes ([8, p. 18]).

IV. $G_7 \subset G_8 \iff \varphi = 0$. This is the group of simply isotropic *boundary* similitudes ([8, p. 8]).

V. $V_7 \subset G_8 \iff c_6c_7^2 = 1$. This is the group of simply isotropic *volume preserving* similitudes ([8, p. 8]).

VI. $G_6 = G_7 \cap V_7$. This is the group of simply isotropic *volume preserving boundary* similitudes ([8, p. 8]).

VII. $B_6 = B_7 \cap G_7$. This is the group of *modular boundary* motions ([8, p. 9]).

VIII. $B_5 = B_7 \cap S_7 \cap G_7$. This is the group of *unimodular boundary* motions ([8, p. 9]).

Basic references on the geometry of the simply isotropic space $I_3^{(1)}$ are Sachs' book [8] and Strubecker's papers [8], [11] and [12].

Using some basic concepts from integral geometry in the sense of R. Deltheil [3], M. I. Stoka [10], G. I. Drinfel'd, and A. V. Lucenko [4], [5], [6], we study the measurability of sets of pairs of intersecting nonisotropic straight lines of type β with respect to G_8 and indicated above subgroups. Analogous problems about sets of pairs of intersecting non-isotropic straight lines of type α in $I_3^{(1)}$ have been treated in [2].

2 Measurability with respect to G_8

Let (G_1, G_2) be a pair of intersecting non-isotropic straight lines of type β . Let G_i have Plücker coordinates (p_j^i) , $i = 1, 2$, $j = 1, \dots, 6$, which satisfy the relations ([8, p. 38])

$$p_1^i p_4^i + p_2^i p_5^i + p_3^i p_6^i = 0, \quad i = 1, 2. \quad (2)$$

Since G_1 and G_2 are intersecting non-isotropic lines of type β , we have

$$p_1^1 p_4^2 + p_2^1 p_5^2 + p_3^1 p_6^2 + p_4^1 p_1^2 + p_5^1 p_2^2 + p_6^1 p_3^2 = 0, \quad p_3^1 - p_3^2 \neq 0, \quad (3)$$

$$|p_1^i| + |p_2^i| \neq 0, \quad i = 1, 2, \quad (4)$$

$$p_1^1 p_2^2 - p_2^1 p_1^2 = 0. \quad (5)$$

Having in mind (4), we can assume, without loss of generality, that $p_1^i = 1$. From (2), p_4^i can be expressed by the remaining Plücker coordinates of G_i , and in view of (3) and (5), p_2^2 and p_6^2 also can be expressed by $p_2^1, p_3^1, p_5^1, p_6^1, p_3^2$ and p_5^2 . Thus the pair (G_1, G_2) can be determined by $p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2$.

Remark 2.1 We note that if G_i , $i = 1, 2$, are represented in the usual way by the equations

$$G_1: \begin{cases} x = a_1(z - r) + p \\ y = b_1(z - r) + q \end{cases}, \quad G_2: \begin{cases} x = a_2(z - r) + p \\ y = \frac{a_2}{a_1} b_1(z - r) + q \end{cases}, \quad (6)$$

where $P(p, q, r) = G_1 \cap G_2$ and $a_1 \neq 0$, $a_2 \neq 0$, then

$$\begin{aligned} p_2^1 &= \frac{b_1}{a_1}, & p_3^1 &= \frac{1}{a_1}, & p_5^1 &= r - \frac{p}{a_1}, & p_6^1 &= p \frac{b_1}{a_1} - q, \\ p_3^2 &= \frac{1}{a_2}, & p_5^2 &= r - \frac{p}{a_2}. \end{aligned} \quad (7)$$

Under the action of (1) the pair $(G_1, G_2)(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$ is transformed into the pair $(\overline{G}_1, \overline{G}_2)(\overline{p}_2^1, \overline{p}_3^1, \overline{p}_5^1, \overline{p}_6^1, \overline{p}_3^2, \overline{p}_5^2)$. Thus we have

$$\begin{aligned} \overline{p}_2^1 &= K c_7 (\sin \varphi + p_2^1 \cos \varphi), \\ \overline{p}_3^1 &= K (c_4 + c_5 p_2^1 + c_6 p_3^1), \\ \overline{p}_5^1 &= K \{ (c_3 - c_5 p_6^1 + c_6 p_5^1) c_7 \cos \varphi \\ &\quad - [c_3 + c_4 p_6^1 + c_6 (p_2^1 p_5^1 + p_3^1 p_6^1)] c_7 \sin \varphi - c_1 (c_4 + c_5 + c_6 p_3^1) \}, \\ \overline{p}_6^1 &= K c_7 [(c_1 p_2^1 - c_2) \cos \varphi + (c_1 + c_2 p_2^1) \sin \varphi + c_7 p_6^1], \\ \overline{p}_3^2 &= K (c_4 + c_5 p_2^2 + c_6 p_3^2), \\ \overline{p}_5^2 &= K \{ (c_3 - c_5 p_6^2 + c_6 p_5^2) c_7 \cos \varphi \\ &\quad - [c_3 + c_4 p_6^2 + c_6 (p_2^2 p_5^2 + p_3^2 p_6^2)] c_7 \sin \varphi - c_1 (c_4 + c_5 + c_6 p_3^2) \}, \end{aligned} \quad (8)$$

where $K = [c_7(\cos \varphi - p_2^1 \sin \varphi)]^{-1}$, $i = 1, 2$. The transformations (8) form the associated group $\overline{G_8}$ of G_8 ([10, p. 34]). The group $\overline{G_8}$ is isomorphic to G_8 and the density with respect to G_8 of the pairs (G_1, G_2) if it exists, coincides with the density with respect to $\overline{G_8}$ of the set of parameters $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$.

The associated group $\overline{G_8}$ has the infinitesimal operators

$$\begin{aligned} X_1 &= p_3^1 \frac{\partial}{\partial p_5^1} - p_2^1 \frac{\partial}{\partial p_6^1} - p_3^2 \frac{\partial}{\partial p_5^2}, & X_2 &= \frac{\partial}{\partial p_6^1}, & X_3 &= \frac{\partial}{\partial p_5^1} + \frac{\partial}{\partial p_5^2}, \\ X_4 &= \frac{\partial}{\partial p_3^1} + \frac{\partial}{\partial p_3^2}, & X_5 &= p_2^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_5^1} + p_2^2 \frac{\partial}{\partial p_3^2} - p_6^2 \frac{\partial}{\partial p_5^2}, \\ X_6 &= p_3^1 \frac{\partial}{\partial p_3^1} + p_5^1 \frac{\partial}{\partial p_5^1} + p_3^2 \frac{\partial}{\partial p_3^2} + p_5^2 \frac{\partial}{\partial p_5^2}, & X_7 &= p_3^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_6^1} + p_3^2 \frac{\partial}{\partial p_3^2}, \\ X_8 &= [1 + (p_2^1)^2] \frac{\partial}{\partial p_2^1} + p_2^1 p_3^1 \frac{\partial}{\partial p_3^1} - p_3^1 p_6^1 \frac{\partial}{\partial p_5^1} + p_2^1 p_6^1 \frac{\partial}{\partial p_6^1} + p_2^2 p_3^2 \frac{\partial}{\partial p_3^2} - p_6^2 p_3^2 \frac{\partial}{\partial p_5^2}, \end{aligned} \quad (9)$$

and it acts transitively on the set of parameters $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$. The infinitesimal operators X_1, X_2, X_3, X_4, X_7 , and X_8 are arcwise unconnected and

$$X_6 = \frac{p_5^2 - p_5^1}{p_3^2 - p_3^1} X_1 + p_6^1 X_2 + \frac{p_3^1 p_5^2 - p_5^1 p_3^2}{p_3^2 - p_3^1} X_3 + X_7.$$

Since

$$X_1 \left(\frac{p_5^2 - p_5^1}{p_3^2 - p_3^1} \right) + X_2(p_6^1) + X_3 \left(\frac{p_3^1 p_5^2 - p_5^1 p_3^2}{p_3^2 - p_3^1} \right) + X_7(1) = 3 \neq 0,$$

we can establish the following

Theorem 2.1 *The set of pairs of intersecting non-isotropic straight lines is not measurable with respect to the group G_8 , and it has no measurable subsets.*

3 Measurability with respect to S_7

The associated group $\overline{S_7}$ of the group S_7 has the infinitesimal operators $X_1, X_2, X_3, X_4, X_5, X_7$, and X_8 from (9), and it acts transitively on the set of parameters $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$. The integral invariant function

$$f = f(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$$

satisfying the so-called system of R. Deltheil (see [3, p. 28]; [10, p. 11])

$$\begin{aligned} X_1(f) &= 0, & X_2(f) &= 0, & X_3(f) &= 0, & X_4(f) &= 0, & X_5(f) &= 0 \\ X_7(f) + f &= 0, & X_8(f) + 5p_2^1 f &= 0 \end{aligned}$$

has the form

$$f = \frac{h}{(p_3^1 - p_3^2)[1 + (p_2^1)^2]},$$

where $h = \text{const.}$

Thus we state the following

Theorem 3.1 *The set of pairs $(G_1, G_2)(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$ is measurable with respect to the group S_7 and has the density*

$$d(G_1, G_2) = \frac{1}{|p_3^2 - p_3^1|[1 + (p_2^1)^2]^2} dp_2^1 \wedge dp_3^1 \wedge dp_5^1 \wedge dp_6^1 \wedge dp_3^2 \wedge dp_5^2. \quad (10)$$

Differentiating (7) and substituting into (10) we obtain other expression for the density:

Corollary 3.1 *The density (10) for the pairs (G_1, G_2) represented by (6) can be written in the form*

$$d(G_1, G_2) = \left| \frac{a_1}{a_2^2(a_1^2 + b_1^2)^2} \right| da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr. \quad (11)$$

4 Some Crofton-type formulas with respect to S_7

Let us consider the isotropic plane ι , which is determined by the lines G_1 and G_2 . The plane ι has the equation

$$\iota: b_1x - a_1y + a_1q - b_1p = 0.$$

If \tilde{P} is the orthogonal projection of P into Oxy , consider the affine coordinate system $(\tilde{P}\vec{e}_1'\vec{e}_2')$ in the isotropic plane ι , where $\vec{e}_1' = (a_1, b_1, 1)$, $\vec{e}_2' = \vec{e}_3$. It should be noticed, that if $\tilde{G} = \iota \cap Oxy$ then $\vec{e}_1' \parallel \tilde{G}$. Let $J^1 = Oxz \cap \iota$ and $J^2 = Oyz \cap \iota$. Obviously

$$J^1: x = p - \frac{a_1}{b_1}q, \quad y = 0, \quad J^2: y = q - \frac{b_1}{a_1}p, \quad x = 0,$$

and J^1, J^2 have the equations

$$J^1: x = -\frac{q}{b_1}, \quad J^2: x = -\frac{p}{a_1}$$

with respect to $(\tilde{P}\vec{e}_1'\vec{e}_2')$.

Then the density $d(J^1, J^2)$ for the pairs (J^1, J^2) with respect to the group H_4^1 , which is the restriction of S_7 into ι , is (see [1, p. 201])

$$d(J^1, J^2) = \left(\frac{p}{a_1} - \frac{q}{b_1} \right)^2 d\frac{p}{a_1} \wedge d\frac{q}{b_1}.$$

Recall that ([8, p. 45])

$$s = \frac{a_1 - a_2}{a_2\sqrt{a_1^2 + b_1^2}} \quad (12)$$

is the angle from G_1 to G_2 , we find

$$d(J^1, J^2) \wedge dP \wedge ds = \frac{(pb_1 - qa_1)pq}{a_1^3b_1^4a_2^2\sqrt{a_1^2 + b_1^2}} da_1 \wedge db_1 \wedge dp \wedge dq \wedge dr \wedge da_2.$$

Comparing with (11), we get

$$d(G_1, G_2) = \left| \frac{a_1^4 b_1^4}{pq(pb_1 - qa_1)(a_1^2 + b_1^2)^{\frac{3}{2}}} \right| d(J^1, J^2) \wedge ds \wedge dP. \quad (13)$$

Let φ_i , $i = 1, 2$, be the angle between G_i and Oxy . Then ([8, p. 48])

$$\varphi_1 = \frac{1}{\sqrt{a_1^2 + b_1^2}}, \quad \varphi_2 = \frac{a_1}{a_2 \sqrt{a_1^2 + b_1^2}}, \quad (14)$$

and (13) becomes

$$d(G_1, G_2) = \left| \frac{a_1^4 b_1^4 \varphi_1^3}{pq(pb_1 - qa_1)} \right| d(J^1, J^2) \wedge ds \wedge dP. \quad (15)$$

By differentiation of (14) and by exterior multiplication by (12), we obtain

$$\begin{aligned} d(G_1, G_2) &= \left| \frac{a_1^4 b_1^4}{pq(pb_1 - qa_1)(a_1^2 + b_1^2)^{\frac{3}{2}}} \right| d(J^1, J^2) \wedge d\varphi_2 \wedge dP \\ &= \left| \frac{a_1^4 b_1^4 \varphi_1^3}{pq(pb_1 - qa_1)} \right| d(J^1, J^2) \wedge d\varphi_2 \wedge dP. \end{aligned} \quad (16)$$

If $\tilde{\varphi}$ is the isotropic distance from J^1 to J^2 , then ([7, p. 19])

$$\tilde{\varphi} = -\frac{p}{a_1} + \frac{q}{b_1}. \quad (17)$$

Putting (17) into (15) and (16), we find

$$d(G_1, G_2) = \left| \frac{a_1^3 b_1^3 \varphi_1^3}{pq\tilde{\varphi}} \right| d(J^1, J^2) \wedge ds \wedge dP = \left| \frac{a_1^3 b_1^3 \varphi_1^3}{pq\tilde{\varphi}} \right| d(J^1, J^2) \wedge d\varphi_2 \wedge dP. \quad (18)$$

Let G_i^1 and G_i^2 be now the projections of G_i into Oxz and Oyz obtained in a parallel way to Oy and Ox , respectively. Then

$$\begin{aligned} G_i^1: \quad z &= \frac{1}{a_i}x + r - \frac{p}{a_i}, \quad y = 0, \quad i = 1, 2, \\ G_1^2: \quad z &= \frac{1}{b_1}y + r - \frac{q}{b_1}, \quad x = 0, \\ G_2^2: \quad z &= \frac{a_1}{a_2 b_1}y + r - \frac{a_1}{a_2 b_1}q, \quad x = 0. \end{aligned}$$

Furthermore,

$$d(G_1^1, G_2^1) = \left| \frac{1}{a_1 a_2 (a_2 - a_1)} \right| da_1 \wedge da_2 \wedge dp \wedge dr \quad (19)$$

is the density for the pairs (G_1^1, G_2^1) in the isotropic plane Oxz with respect ${}^1H_4^1$ which is the restriction of S_7 into Oxz and

$$d(G_1^2, G_2^2) = \left| \frac{1}{b_1^2 a_2 (a_2 - a_1)} \right| (a_1 db_1 \wedge da_2 - a_2 db_1 \wedge da_1) \wedge dq \wedge dr$$

is the density for the pairs (G_1^2, G_2^2) in the isotropic plane Oyz with respect ${}^2H_4^1$ which is the restriction of S_7 into Oyz (see [1, p. 177]).

By exterior multiplication of (G_1^1, G_2^1) and $ds \wedge dq$, we get

$$d(G_1, G_2) = \left| \frac{a_1^2 \varphi_1}{b_1} \right| d(G_1^1, G_2^1) \wedge ds \wedge dq, \quad (20)$$

and by exterior multiplication of (19) and $d\varphi_1 \wedge dq$:

$$d(G_1, G_2) = \left| \frac{a_1^2 s}{b_1} \right| d(G_1^1, G_2^1) \wedge d\varphi_1 \wedge dq. \quad (21)$$

If, instead of using $d\varphi_1 \wedge dq$, we multiply by $d\varphi_2 \wedge dq$, we obtain

$$d(G_1, G_2) = \left| \frac{a_1 a_2 s}{b_1} \right| d(G_1^1, G_2^1) \wedge d\varphi_2 \wedge dq. \quad (22)$$

Analogously, we can derive the following formulas:

$$\begin{aligned} d(G_1, G_2) &= \left| \frac{a_1^2 b_1^2 \varphi_1}{a_2^3} \right| d(G_1^2, G_2^2) \wedge ds \wedge dp \\ &= \left| \frac{b_1^2 s}{a_1} \right| d(G_1^2, G_2^2) \wedge d\varphi_1 \wedge dp \\ &= \left| \frac{a_2 b_1^2 s}{a_1^2} \right| d(G_1^2, G_2^2) \wedge d\varphi_2 \wedge dp. \end{aligned} \quad (23)$$

In summary, the following theorem holds.

Theorem 4.1 *The density for the set of pairs (G_1, G_2) of intersecting non-isotropic straight lines of type β , determined by (6), with respect to the group S_7 satisfies the relations (15), (16), (18), (20), (21), (22), and (23).*

5 Measurability with respect to G_6

Now, the corresponding associated group $\overline{G_6}$ has the infinitesimal operators

$$\begin{aligned} Y_1 &= p_3^1 \frac{\partial}{\partial p_5^1} - p_2^1 \frac{\partial}{\partial p_6^1} + p_3^2 \frac{\partial}{\partial p_5^2}, & Y_2 &= \frac{\partial}{\partial p_6^1}, \\ Y_3 &= \frac{\partial}{\partial p_5^1} + \frac{\partial}{\partial p_5^2}, & Y_4 &= p_2^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_5^1} + p_2^1 \frac{\partial}{\partial p_3^2} - p_6^1 \frac{\partial}{\partial p_5^2}, \\ Y_7 &= 3p_3^1 \frac{\partial}{\partial p_3^1} + 2p_5^1 \frac{\partial}{\partial p_5^1} - p_6^1 \frac{\partial}{\partial p_6^1} + 3p_3^2 \frac{\partial}{\partial p_3^2} + 2p_5^2 \frac{\partial}{\partial p_5^2}, & Y_8 &= \frac{\partial}{\partial p_1^1} + \frac{\partial}{\partial p_3^2}. \end{aligned}$$

The group $\overline{G_6}$ acts intransitively on the set of points $(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$ and therefore the set of pairs (G_1, G_2) has not invariant density with respect to G_6 . The system

$$Y_1(f) = 0, Y_2(f) = 0, Y_3(f) = 0, Y_4(f) = 0, Y_7(f) = 0, Y_8(f) = 0$$

has the solution

$$f = p_2^1,$$

and it is an absolute invariant of G_6 . Consider the subset of pairs (G_1, G_2) satisfying the condition

$$p_2^1 = h, \quad (24)$$

where $h = \text{const}$. The group $\overline{G_6}$ induces on this subset the group G_6^* with the infinitesimal operators

$$\begin{aligned} Z_1 &= p_3^1 \frac{\partial}{\partial p_5^1} - h \frac{\partial}{\partial p_6^1} + p_3^2 \frac{\partial}{\partial p_5^2}, & Z_2 &= \frac{\partial}{\partial p_6^1}, \\ Z_3 &= \frac{\partial}{\partial p_5^1} + \frac{\partial}{\partial p_5^2}, & Z_4 &= p_2^1 \frac{\partial}{\partial p_3^1} - p_6^1 \frac{\partial}{\partial p_5^1} + p_2^1 \frac{\partial}{\partial p_3^2} - p_6^1 \frac{\partial}{\partial p_5^2}, \\ Z_7 &= 3p_3^1 \frac{\partial}{\partial p_3^1} + 2p_5^1 \frac{\partial}{\partial p_5^1} - p_6^1 \frac{\partial}{\partial p_6^1} + 3p_3^2 \frac{\partial}{\partial p_3^2} + 2p_5^2 \frac{\partial}{\partial p_5^2}, & Z_8 &= \frac{\partial}{\partial p_1^1} + \frac{\partial}{\partial p_3^2}. \end{aligned}$$

The integral invariant function $f = f(p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$, which satisfies the Deltheil system

$$Z_1(f) = 0, \quad Z_2(f) = 0, \quad Z_3(f) = 0, \quad Z_4(f) = 0, \quad Z_7(f) - 9f = 0, \quad Z_8(f) = 0,$$

has the form

$$f = \frac{c}{(p_3^1 - p_3^2)^3},$$

where $c = \text{const}$.

Thus we state the following

Theorem 5.1 *The set of pairs $(G_1, G_2)(p_2^1, p_3^1, p_5^1, p_6^1, p_3^2, p_5^2)$ of intersecting non-isotropic lines of type β is not measurable with respect to G_6 , but it has the measurable subset*

$$p_2^1 = h, \quad h = \text{const},$$

with the density

$$d(G_1, G_2) = \frac{1}{|p_3^2 - p_3^1|^3} dp_3^1 \wedge dp_5^1 \wedge dp_6^1 \wedge dp_3^2 \wedge dp_5^2. \quad (25)$$

Differentiating (7), (24), and replacing into (25), we establish

Corollary 5.1 *The set of pairs (G_1, G_2) of intersecting non-isotropic lines of type β , determined by (6), is not measurable with respect to the group G_6 , but it has the measurable subset*

$$\frac{b_1}{a_1} = h, \quad h = \text{const},$$

with the density

$$d(G_1, G_2) = \frac{1}{(a_1 - a_2)^2} da_1 \wedge da_2 \wedge dp \wedge dq \wedge dr.$$

6 Measurability with respect to B_7 , W_7 , G_7 , V_7 , B_6 , and B_5

By arguments similar to those used in the sections 2, 3, and 5, we investigated the measurability with respect to all the remaining groups. We have the following results:

Theorem 6.1 *The set of pairs (G_1, G_2) of intersecting non-isotropic straight lines of type β , determined by (6), is measurable with respect to the group*

(i) B_7 and it has the density

$$d(G_1, G_2) = \left| \frac{a_1 a_2}{(a_1 - a_2)^3 \sqrt{a_1^2 + b_1^2}} \right| da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr;$$

(ii) V_7 and it has the density

$$d(G_1, G_2) = \frac{|a_1|}{(a_1 - a_2)^2 (a_1^2 + b_1^2)} da_1 \wedge db_1 \wedge da_2 \wedge dp \wedge dq \wedge dr.$$

Theorem 6.2 *With respect to the groups W_7 and S_7 the set of pairs (G_1, G_2) of intersecting non-isotropic lines of type β is not measurable and it has no measurable subsets.*

Theorem 6.3 *The set of pairs (G_1, G_2) of intersecting non-isotropic straight lines of type β , determined by (6), is not measurable with respect to the group*

(i) B_6 , but it has the measurable subset

$$\frac{b_1}{a_1} = h, \quad h = \text{const},$$

with the density

$$d(G_1, G_2) = \left| \frac{a_1 a_2}{(a_1 - a_2)^3} \right| da_1 \wedge da_2 \wedge dp \wedge dq \wedge dr;$$

(ii) B_5 , but it has the measurable subset

$$\frac{b_1}{a_1} = h_1, \quad \frac{1}{a_1} - \frac{1}{a_2} = h_2, \quad h_1, h_2 = \text{const},$$

with the density

$$d(G_1, G_2) = \left| \frac{a_2}{a_1(a_1 - a_2)} \right| da_1 \wedge da_2 \wedge dp \wedge dq \wedge dr.$$

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