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Second-order Sufficient Condition for $\tilde{\ell}$-stable Functions*

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Abstract

The aim of our article is to present a proof of the existence of local minimizer in the classical optimality problem without constraints under weaker assumptions in comparisons with common statements of the result. In addition we will provide rather elementary and self-contained proof of that result.

Key words: Second-order derivative; $C^{1,1}$ function; stable function; isolated minimizer of order 2.

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1 Introduction

Past all doubt it is very important to be able to find the minimum or maximum of a function. Recall for example that by J. von Neumann every physical system tends to have its minimum of internal energy.

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From the basic course of mathematical analysis we know the second-order condition for strict local minimum (see [Zo]) given in Theorem 1 below. We will use the following notation and terminology.

We denote by $f'(x;h)$, i.e.

$$f'(x;h) = \lim_{t\downarrow 0} \frac{f(x+th)-f(x)}{t},$$

the first-order directional derivative of $f:\mathbb{R}^N \to \mathbb{R}$ at $x \in \mathbb{R}^N$ in direction $h \in \mathbb{R}^N$

If there exists $f'(x) \in \mathcal{L}(\mathbb{R}^N, \mathbb{R})$ (it means $f'(x)$ is an element of the set of all continuous linear mappings from $\mathbb{R}^N$ to $\mathbb{R}$) such that $f'(x)h = f'(x;h)$ for every $h \in S^N_{\mathbb{R}} = \{y \in \mathbb{R}^N; \|y\| = 1\}$, and the limit in the definition of $f'(x)h$ is uniform for $h \in S^N_{\mathbb{R}}$, then we say that $f:\mathbb{R}^N \to \mathbb{R}$ is *Fréchet differentiable at* $x \in \mathbb{R}^N$.

Further,

$$f''(x;u,v) = \lim_{t\downarrow 0} \frac{f'(x+tu;v)-f'(x;v)}{t}$$

denotes the second-order directional derivative of f at x in direction $(u,v) \in \mathbb{R}^N$.

We say that $f:\mathbb{R}^N \to \mathbb{R}$ is a C^2 function near $x \in \mathbb{R}^N$ if it has continuous second-order partial derivatives on some neighbourhood of x.

Analogously, we will say that a function $f:\mathbb{R}^N \to \mathbb{R}$ satisfies a p-property near $x \in \mathbb{R}^N$ if that p-property holds on some neighbourhood of x.

Recall that $x \in \mathbb{R}^N$ is an isolated minimizer of order 2 for a function $f:\mathbb{R}^N \to \mathbb{R}$ if there are a neighbourhood U of x and $A > 0$ satisfying $f(y) \geq f(x) + A\|y-x\|^2$ for every $y \in U$. We notice that each isolated minimizer of order 2 is a strict local minimizer.

Theorem 1 *Let $f:\mathbb{R}^N \to \mathbb{R}$ be a C^2 function near $x \in \mathbb{R}^N$. If $f'(x) = 0$, and*

$$f''(x;h,h) > 0,$$

for every $h \in S^N_{\mathbb{R}}$, then x is an isolated minimizer of order 2 for f.

Since in some problems of applied mathematics—as for example in variational inequalities, semi-infinite programming, penalty functions, proximal point methods, iterated local minimization by decomposition or augmented Lagrangian—differentiable functions which are not twice differentiable appear (see e.g. [HSN, KT, TR, Q1, Q2]), it was studied the following class of functions.

We say that $f:\mathbb{R}^N \to \mathbb{R}$ is a $C^{1,1}$ *function* near $x \in \mathbb{R}^N$ if it is differentiable on some neighbourhood of x and its derivative $f'(\cdot)$ is Lipschitz there.

It is clear that the class of $C^{1,1}$ functions includes the class of C^2 functions. On the other hand, considering a function $f:\mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \int_0^x |t|dt, \quad \forall x \in \mathbb{R},$$

we have that $f'(x) = |x|$, for every $x \in \mathbb{R}$. It means that $f'(x)$ is Lipschitz function on $\mathbb{R}$, but f is not twice differentiable at 0.

R. Cominetti and R. Correa generalized Theorem 1 by the following way in 1990.

Theorem 2 [CC] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^N$. If $f'(x) = 0$, and*

$$f_\infty(x; h) := \liminf_{y \to x, t \downarrow 0} \frac{f'(y + th; h) - f'(y; h)}{t} > 0,$$

for every $h \in S_{\mathbb{R}}^N$, then x is an isolated minimizer of order 2 for f.

The second-order condition from Theorem 2 was improved by elimination of strict convergence in 2004. We used a certain derivative of the Dini type.

Theorem 3 [BP1] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^N$. If $f'(x) = 0$, and*

$$f_D''^{\ell}(x; h) := \liminf_{t \downarrow 0} \frac{f'(x + th; h) - f'(x; h)}{t} > 0,$$

for every $h \in S_{\mathbb{R}}^N$, then x is an isolated minimizer of order 2 for f.

I. Ginchev, A. Guerraggio and M. Rocca presented the generalization of Theorem 1 in terms of the Peano derivative in 2006.

Theorem 4 [GGR] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a $C^{1,1}$ function near $x \in \mathbb{R}^N$. If $f'(x) = 0$, and*

$$f_P''^{\ell}(x; h) := \liminf_{t \downarrow 0} \frac{f(x + th) - f(x) - tf'(x; h)}{t^2/2} > 0,$$

for every $h \in S_{\mathbb{R}}^N$, then x is an isolated minimizer of order 2 for f.

It can be easily derived from [TR, Theorem 4] that $f_P''^{\ell}(x; h) \geq f_D''^{\ell}(x; h)$. Moreover, since the calculus with $f_P''^{\ell}(x; h)$ seems to be more comfortable than that with $f_D''^{\ell}(x; h)$, we can say that Theorem 3 lost its sense after Theorem 4. Example 1 confirms this fact.

Example 1 Let us consider a function

$$f(x) = \begin{cases} \int_0^{|x|} t(\frac{19}{20} + \sin \ln t)\, dt, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

In [BP2], we showed that f is a $C^{1,1}$ function, $f_D''^{\ell}(0; 1) = \frac{19}{20} - 1 < 0$, and $f_P''^{\ell}(0; 1) = f_P''^{\ell}(0; -1) = \frac{19}{20} + \frac{2}{5}(-\sqrt{5}) > 0$. Due to Theorem 4 the function f attains its strict local minimum, but Theorem 3 is not applicable.

Another result was stated by A. Ben-Tal and J. Zowe in 1985. A function $f : \mathbb{R}^N \to \mathbb{R}$ for which there exist a neighbourhood U of $x \in \mathbb{R}^N$ and $K > 0$ such that for all $y \in U$ there exists the Fréchet derivative $f'(y)$ and

$$\|f'(y) - f'(x)\| \le K\|y - x\|, \quad \forall y \in U,$$

is called *stable at* x. We note that if $f : \mathbb{R}^N \to \mathbb{R}$ is a $C^{1,1}$ function near $x \in \mathbb{R}^N$, then f is stable at x.

Theorem 5 [BZ] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be Fréchet differentiable near $x \in \mathbb{R}^N$ and let f be stable at x. If $f'(x) = 0$ and*

$$f''_P(x; h) := \lim_{t\downarrow 0} \frac{f(x + th) - f(x) - tf'(x; h)}{t^2/2} > 0,$$

for every $h \in S^N_{\mathbb{R}}$, then x is a strict local minimizer 2 for f.

Finally, we note that we generalized both Theorems 4, 5 in terms of so called ℓ–stable functions as follows.

We say that a function $f : \mathbb{R}^N \to \mathbb{R}$ is *ℓ-stable at* $x \in \mathbb{R}^N$ if there exist a neighbourhood U of x and $K > 0$ such that

$$|f^\ell(y; h) - f^\ell(x; h)| \le K\|y - x\|, \quad \forall y \in U, \forall h \in S^N_{\mathbb{R}},$$

where

$$f^\ell(x; h) = \liminf_{t\downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

It is worth to note that the ℓ–stability at x implies the strict differentiability of the function at the point x.

Theorem 6 [BP2] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be continuous near $x \in \mathbb{R}^N$ and let f be ℓ-stable at x. If $f'(x) = 0$, and*

$$f'^{\ell}_P(x; h) > 0,$$

then x is an isolated minimizer of order 2 for f.

The $C^{1,1}$ property can be generalized also in the following way.

Definition 1 We say that $f : \mathbb{R}^N \to \mathbb{R}$ is $\tilde{\ell}$-stable at $x \in \mathbb{R}^N$ if there exist a neighbourhood U of x and $K > 0$ such that

$$|f^\ell(z; z - y) - f^\ell(y; z - y)| \le K\|z - y\|^2, \quad \forall y, z \in U.$$

Remark 1 Notice that a function $f : \mathbb{R}^N \to \mathbb{R}$ is $\tilde{\ell}$-stable at $x \in \mathbb{R}^N$ if there exist a neighbourhood U of x and $K > 0$ such that

$$|f^\ell(y + th; h) - f^\ell(y; h)| \le Kt,$$

for every $h \in S^N_{\mathbb{R}}$, $y \in U$ and $t > 0$ satisfying $y + th \in U$.

Remark 2 Notice that verifying the ℓ-stability we compare $f^\ell(\cdot,\cdot)$ for points from U only with x but in all directions. Conversely, verifying the $\tilde{\ell}$-stability we compare $f^\ell(\cdot,\cdot)$ for every point from U with every point from U but only in the corresponding direction.

In [BP3], we passed the problem whether we can replace the condition to be ℓ-stable by the condition to be $\tilde{\ell}$-stable in Theorem 6. In Section 3, we will answer this question in the affirmative. Before it, in Section 2, we will examine some properties of $\tilde{\ell}$-stable functions.

2 $\tilde{\ell}$-stability

At first, we will derive that the $\tilde{\ell}$-stability together with continuity implies the Lipschitzness.

Lemma 1 [BP2, Lemma 4] *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function, and let $a, b \in \mathbb{R}^N$. Then there exist $\xi_1, \xi_2 \in (a, b)$ such that*

$$f^\ell(\xi_1; b-a) \le f(b) - f(a) \le f^\ell(\xi_2; b-a). \tag{1}$$

Lemma 2 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function near $x \in \mathbb{R}^N$ and let f be $\tilde{\ell}$–stable at x. Then there exists a neighbourhood V of x such that*

$$\sup_{h \in S^N_{\mathbb{R}},\ y \in V} |f^\ell(y; h)| < \infty.$$

Proof Suppose on the contrary that there are sequences $\{y_n\}_{n=1}^\infty \subset \mathbb{R}^N$, $\{h_n\}_{n=1}^\infty \subset S^N_{\mathbb{R}}$ such that $y_n \to x$ as $n \to \infty$ and

$$\lim_{n\to\infty} |f^\ell(y_n; h_n)| = \infty.$$

Without any loss of generality we can assume that either

$$\lim_{n\to\infty} f^\ell(y_n, h_n) = -\infty$$

or

$$\lim_{n\to\infty} f^\ell(y_n, h_n) = +\infty.$$

We suppose that the first case occurs (the second case can be treated by an analogous way).

Next we can assume that for certain $\gamma > 0$ the condition in Definition 1 of the ℓ-stability is fulfilled on $B(x, \gamma)$, and moreover f is continuous and bounded on $B(x, \gamma)$. Let $\delta > 0$ denote a constant such that for each sufficiently large $n \in \mathbb{N}$ we have : $y_n + \delta h_n \in B(x, \gamma)$.

Now, if we combine the $\tilde{\ell}$-stability and Lemma 1, for each sufficiently large $n \in \mathbb{N}$ we get $\xi_n \in (y_n, y_n + \delta h_n)$ such that the following holds :

$$\begin{aligned} f(y_n + \delta h_n) &\le f(y_n) + \delta f^\ell(\xi_n; h_n) \\ &= f(y_n) + \delta[f^\ell(\xi_n; h_n) - f^\ell(y_n + \delta h_n; h_n) + f^\ell(y_n + \delta h_n; h_n) \\ &\quad - f^\ell(y_n; h_n) + f^\ell(y_n; h_n)] \\ &\le f(y_n) + 2K\delta^2 + \delta f^\ell(y_n; h_n). \end{aligned}$$

Since f is bounded on $B(x,\gamma)$ and $f^\ell(y_n;h_n) \to -\infty$, the previous inequality does not hold for any sufficiently large $n \in \mathbb{N}$,which is a contradiction. □

Proposition 1 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function near $x \in \mathbb{R}^N$ and let f be $\tilde{\ell}$-stable at x. Then f is Lipschitz near x.*

Proof Due to Lemma 2 there exists a ball $B(x,\delta)$ on which f is continuous and

$$L := \sup_{y\in B(x,\delta),h\in S_{\mathbb{R}^N}} |f^\ell(y;h)| < \infty.$$

Next by Lemma 1, for any pair of points $a,b \in B(x,\delta)$ there exists $\xi \in (a,b) \subset B(x,\delta)$ such that

$$|f(b)-f(a)| \le |f^\ell(\xi;(b-a)/\|b-a\|)|\|b-a\| \le L\|b-a\|.$$ □

Now, we will show some properties of $\tilde{\ell}$-stable functions concerning differentiability.

Lemma 3 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be L-Lipschitz near $x \in \mathbb{R}^N$. Then*

$$|f^\ell(x;h_2) - f^\ell(x;h_1)| \le L\|h_1-h_2\|, \quad \forall h_1,h_2 \in \mathbb{R}^N.$$

Proof We consider arbitrary $h_1,h_2 \in \mathbb{R}^N$. For sufficiently small $t>0$ it holds

$$\begin{aligned} -L\|h_2-h_1\| &\le \frac{f(x+th_2)-f(x+th_1)}{t} \\ &= \frac{f(x+th_2)-f(x)}{t} - \frac{f(x+th_1)-f(x)}{t} \\ &= \frac{f(x+th_2)-f(x+th_1)}{t} \le L\|h_2-h_1\|. \end{aligned}$$

Hence,

$$|f^\ell(x;h_2)-f^\ell(x;h_1)| \le L\|h_2-h_1\|.$$ □

Proposition 2 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function near $x \in \mathbb{R}^N$ and let f be $\tilde{\ell}$-stable at x. Then for every y sufficiently close to x we have*

(1) f is directionally differentiable at y and $f'(y;-h) = -f'(y;h)$ for every $h \in S^N_{\mathbb{R}}$.

(2) the mapping $h \mapsto f'(y;h)$ from $\mathbb{R}^N$ to $\mathbb{R}$ is Lipschitz.

Proof Assume that f is continous on some neighborhood U of x and that the $\tilde{\ell}$-stability property holds on U, too. This means, that for some $K>0$, we have:

$$|f^\ell(z;z-y) - f^\ell(y;z-y)| \le K\|z-y\|^2, \tag{2}$$

for every $z, y \in U$. Now fix $y_0 \in U$, $h_0 \in S^N_{\mathbb{R}}$ and show the existence of the directional derivative $f'(y_0; h_0)$. To do this, we will employ the following auxiliary function $g : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$g(t) = f(y_0 + th_0), \quad t \in \mathbb{R}.$$

Now we can express its lower Dini directional derivative in the following form $g^\ell(t; 1) = f^\ell(y_0 + th_0; h_0)$. We have in particular that $g^\ell(0; 1) = f^\ell(y_0; h_0)$. Now we will show that the function g is $\tilde{\ell}$-stable at zero. So let us consider two arbitrary points $t_1, t_2 \in \mathbb{R}$ such that $y_0 + t_i h_0 \in U$, $i = 1, 2$. Then by (2) we have:

$$\begin{aligned} &|g^\ell(t_1; t_1 - t_2) - g^\ell(t_2; t_1 - t_2)| \\ = &|f^\ell(y_0 + t_1 h_0; (t_1 - t_2)h_0) - f^\ell(y_0 + t_2 h_0; (t_1 - t_2)h_0)| \le K|t_1 - t_2|^2. \end{aligned} \tag{3}$$

Thus g is $\tilde{\ell}$-stable at $t = 0$. Now by the lipschitzness of f near x the function g must be Lipschitz near $t = 0$. Hence, from the Rademacher theorem it follows the existence of a sequence $\{t_n\}_{n=1}^\infty$ such that $t_n \downarrow 0$, and for every $n \in \mathbb{N}$ there exists $g'(t_n) \in \mathbb{R}$. By (3) the sequence $\{g'(t_n)\}_{n=1}^\infty$ is Cauchy and consequently there exists a limit

$$L = \lim_{n\to\infty} g'(t_n) \in \mathcal{L}(\mathbb{R}, \mathbb{R}). \tag{4}$$

In what follows, we will show that in fact $L = f'(y_0; h_0)$. This will be true if we prove that for each sequence $\{s_k\}_{k=1}^\infty$ such that $s_k \downarrow 0$ it holds

$$\left| L - \frac{f(y_0 + s_k h_0) - f(y_0)}{s_k} \right| \to 0 \quad \text{as } k \to \infty.$$

Now for every $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that $t_{n_k} \in (0, s_k)$ and furthermore, by Lemma 1 and (3) there are $\xi_k, \xi'_k \in (0, s_k)$ such that

$$\begin{aligned} K|t_{n_k} - \xi_k| &\le g'(t_{n_k}) - g^\ell(\xi_k; 1) \le g'(t_{n_k}) - \frac{g(s_k) - g(0)}{s_k} \\ &\le g'(t_{n_k}) - g^\ell(\xi'_k; 1) \le K|t_{n_k} - \xi'_k|. \end{aligned}$$

This immediately implies that

$$\left| g'(t_{n_k}) - \frac{g(s_k) - g(0)}{s_k} \right| \to 0 \quad \text{as } k \to \infty. \tag{5}$$

Now since for every $k \in \mathbb{N}$ we have:

$$\left| L - \frac{f(y_0 + s_k h_0) - f(y_0)}{s_k} \right| \le |L - g'(t_{n_k})| + \left| g'(t_{n_k}) - \frac{g(s_k) - g(0)}{s_k} \right|,$$

by (4), (5) we get the existence of limit

$$L = \lim_{k\to\infty} \frac{f(y_0 + s_k h_0) - f(x)}{s_k},$$

whenever $\{s_k\}_{k=1}^{\infty}$ is a sequence such that $s_k \downarrow 0$. Hence the following limit exists:

$$L = \lim_{s \downarrow 0} \frac{f(y_0 + sh_0) - f(y_0)}{s}.$$

The assertion (2) now follows immediately from Proposition 1 and Lemma 3. □

3 Main result

Theorem 7 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be continuous near $x \in \mathbb{R}^N$ and let f be $\tilde{\ell}$-stable at x. If $f'(x; h) = 0$ for all $h \in S_{\mathbb{R}^N}$, and*

$$f'^{\ell}_P(x; h) > 0, \quad \forall h \in S_{\mathbb{R}^N},$$

then x is an isolated minimizer of order 2 for f.

Proof Without loss of generality we can assume that $x = 0$, and $f(0) = 0$. In the proof, we will apply the mathematical induction on the dimension N. First put $N = 1$ and suppose on the contrary that the assertion does not hold. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \to 0$ as $n \to \infty$, and

$$f(x_n) \le \frac{1}{n}|x_n|^2, \quad \forall n \in \mathbb{N}. \tag{6}$$

Suppose, for example, that $x_n > 0$ for every $n \in \mathbb{N}$. By the hypothesis of theorem it follows that there are $\delta > 0$, $\alpha > 0$ such that

$$\frac{f(t \cdot 1)}{t^2} \ge \alpha > 0, \quad \forall t \in (0, \delta). \tag{7}$$

By (6) and (7) we have for $n \in \mathbb{N}$ sufficiently large, that

$$\alpha \le \frac{f(x_n)}{x_n^2} \le \frac{1}{n},$$

hence a contradiction. Thus the assertion is true for $N = 1$.

Now let the assertion holds for $N \ge 1$ and we will prove it for $N + 1$. To do this, let us assume again that $\hat{x} = 0$ is not a minimizer of order 2 for f. This implies the existence of some sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{N+1}$ such that $x_n \to 0$ as $n \to \infty$, and

$$f(x_n) \le \frac{1}{n}\|x_n\|^2, \quad \forall n \in \mathbb{N}. \tag{8}$$

Without loss of generality it can be assumed that for some neighbourhood $U(0)$ of zero, $x_n \in U(0)$ for every $n \in \mathbb{N}$, and on $U(0)$ the $\tilde{\ell}$-stability is valid. By the compactness of the unit sphere we can assume that for some $h_0 \in S_{N+1}$, $h_n := x_n/\|x_n\| \to h_0$ as $n \to \infty$. First suppose that for infinitely many $n \in \mathbb{N}$, x_n are contained in some linear subspace $L \subset \mathbb{R}^{N+1}$ of dimension $k \le N$. Then, according to our induction assumption, $\hat{x} = 0$ is an isolated minimizer od order 2 for f which contradicts the property (8). Now let us suppose that this is not the case. Further let $0 < \rho < 1 - \frac{\sqrt{2}}{2}$ and let $v_1, \ldots, v_{N+1} \in S_{\mathbb{R}^{N+1}} \cap B(h_0; \rho)$ are linearly independent vectors generating a convex cone C with nonempty interior. Without loss of generality we can assume that $x_n \in \operatorname{int} C$, for every $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed and $t_n = \|x_n\|$. Let F_i be the i-th boundary face of C such that

$$F_i \cap \{x_n + s(h_n - h_0) : s \geq 0\} = \{c_n\}. \tag{9}$$

Then $c_n/\|c_n\| \in S_{\mathbb{R}^{N+1}} \cap B(h_0; \rho)$. In view of our induction assumption, there exist some neighbourhood $V(0)$ and $A > 0$ such that:

$$\frac{f(c)}{\|c\|^2} \geq A > 0, \tag{10}$$

for every $c \in V(0) \cap \partial C$, where ∂C denotes the boundary of C. Further for some $\delta_0 > 0$, we have:

$$\frac{f(th_0)}{t^2} \geq A > 0, \quad \forall t \in (0, \delta_0). \tag{11}$$

From (8) and (10) it follows that if n is sufficiently large, then

$$\frac{f(c_n)}{t_n^2} \geq \frac{f(c_n)}{\|c_n\|^2} \geq A > \frac{1}{n} \geq \frac{f(x_n)}{t_n^2},$$

where $t_n := \|x_n\|$ and $\|c_n\| \geq t_n$. The last inequality can be shown as follows: $c_n = x_n + s_n(h_n - h_0)$, $s_n \geq 0 \Rightarrow c_n = (t_n + s_n)h_n - s_n h_0 \Rightarrow \|c_n\| = \|(t_n + s_n)h_n - s_n h_0\| \geq \|(t_n + s_n)h_n\| - \|s_n h_0\| = (t_n + s_n) - s_n = t_n$. Hence $\|c_n\| \geq t_n$. The last argument shows that $f(c_n) > f(x_n)$ if n is large enough. Next, Lemma 1 gives $\eta_n \in (c_n, x_n)$ such that

$$f^{\ell}(\eta_n; x_n - c_n) \leq f(x_n) - f(c_n) < 0. \tag{12}$$

Now again using Lemma 1, (8), (11), and (12), we have that for some $\xi_n \in (t_n h_n, t_n h_0)$ and n large enough, it holds:

$$\begin{aligned} 0 < \frac{A}{2} \leq \frac{f(t_n h_0) - f(t_n h_n)}{t_n^2} &\leq \frac{f^{\ell}(\xi_n; t_n(h_0 - h_n))}{t_n^2} \\ < \frac{f^{\ell}(\xi_n; h_0 - h_n) - f^{\ell}(\eta_n; h_0 - h_n)}{t_n}&. \end{aligned} \tag{13}$$

Claim 1 *If $n \in \mathbb{N}$ is large enough, then $\|\xi_n - \eta_n\| < t_n$.*

Proof We will now express the difference $\xi_n - \eta_n$:

$$\begin{aligned} \xi_n - \eta_n &= t_n h_n + \theta_1(t_n h_0 - t_n h_n) - [c_n + \theta_2(x_n - c_n)] \\ &= (\theta_1 t_n + s_n - \theta_2 s_n)(h_0 - h_n), \end{aligned}$$

where $\theta_1, \theta_2 \in (0,1),\ s_n \geq 0$ so that $c_n = x_n + s_n(h_n - h_0)$. For a two-dimensional picture see the Figure 1.

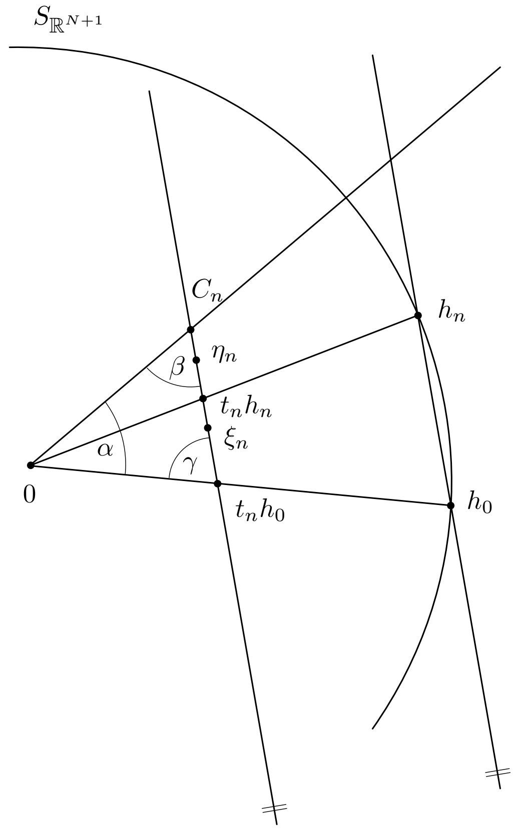

Fig. 1: Two-dimensional picture

Let us consider a triangle with vertices $0, t_n h_0, c_n$ and denote its inner angles by α, β, γ, respectively. It is clear that $\gamma \leq \pi/2$ and $\alpha + \beta + \gamma = \pi$. This implies:

$$\beta = \pi - \alpha - \gamma \geq \frac{\pi}{2} - \alpha. \tag{14}$$

Now let $v_i,\ i = 1 \ldots, N+1$, be one of the vectors generating the cone C and let φ denotes the angle between vectors h_0 and v_i, respectively. Then we have:

$$\begin{aligned}\cos\varphi &= \langle h_0, v_i\rangle = \langle h_0, v_i - h_0 + h_0\rangle = \langle h_0, v_i - h_0\rangle + \langle h_0, h_0\rangle \\ &= 1 + \langle h_0, v_i - h_0\rangle \geq 1 - \|v_i - h_0\| \geq 1 - \rho > 1 - \left(1 - \frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{2} \Rightarrow \varphi < \frac{\pi}{4}.\end{aligned}$$

Suppose that as in the picture, α denotes the angle between vectors h_0 and $c_n/\|c_n\|$. Then also $\alpha < \pi/4$. Indeed, since we can express $c_n/\|c_n\|$ as a convex combination of the pair of vectors v_i, v_{i+1}, $i \in \{1, \dots, N\} : c_n/\|c_n\| = \lambda v_i + (1-\lambda)v_{i+1}$, where $\lambda \in [0,1]$, we have

$$\begin{aligned}\cos\alpha &= \left\langle h_0, \frac{c_n}{\|c_n\|} \right\rangle = \langle h_0, \lambda v_i + (1-\lambda)v_{i+1}\rangle \\ &= \lambda\langle h_0, v_i\rangle + (1-\lambda)\langle h_0, v_{i+1}\rangle > \frac{\sqrt{2}}{2} \Rightarrow \alpha < \frac{\pi}{4}.\end{aligned}$$

Thus, we have proved that for the choice of $\rho \in (0, 1-\sqrt{2}/2)$, we have the right estimation for the angle α. Now by (14) it holds: $\beta \geq \pi/4 > \alpha$. This implies the following property of the lengths of sides of the triangle:

$$t_n > \|c_n - t_n h_0\| > \|\xi_n - \eta_n\|.$$

Thus we proved Claim. □

So we are now able to finish the proof of Theorem 7. If $n \in \mathbb{N}$ is large enough, then following (12) and (13), by the $\tilde{\ell}$-stability and by Claim, we can write

$$\begin{aligned}0 &< \frac{A}{2} < \frac{f^\ell(\xi_n; h_0 - h_n) - f^\ell(\eta_n; h_0 - h_n)}{t_n} \\ &= \frac{1}{\sigma}\frac{f^\ell(\xi_n; \xi_n - \eta_n) - f^\ell(\eta_n; \xi_n - \eta_n)}{t_n} \leq \frac{1}{\sigma}\frac{K\|\xi_n - \eta_n\|^2}{t_n} \\ &< \frac{1}{\sigma}K\|\xi_n - \eta_n\| = K\|h_n - h_0\|, \qquad (15)\end{aligned}$$

where $\sigma := \theta_1 t_n + s_n - \theta_2 s_n > 0$ (see the proof of the claim). Now since $\|h_n - h_0\| \to 0$ as $n \to \infty$, we get a contradiction. □

4 Final remarks and questions

There exist functions which are ℓ-stable but not $\tilde{\ell}$-stable at some point. See e.g. [BP2, Ex.2]. It is not clear whether there exists a function which is $\tilde{\ell}$-stable but not ℓ–stable at a certain point. Also note, that the $C^{1,1}$ property implies in an obvious way the $\tilde{\ell}$-stability and $\tilde{\ell}$-stability. This means that Theorem 7 covers the above mentioned theorems 3 and 4 as well as [LK,Theorem 3.4].

Now it seems natural to ask the following questions.

Question 1 Can we distinguish $C^{1,1}$ functions near x and $\tilde{\ell}$-stable functions at x; or can be the $\tilde{\ell}$-stability at x a characterization of $C^{1,1}$ property near x or not?

Theorem 7 would be more elegant if one could answer the following question in the affirmative.

Question 2 Does the the $\tilde{\ell}$-stability of $f : \mathbb{R}^N \to \mathbb{R}$ at $x \in \mathbb{R}^N$ imply the continuity of f near x?

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